21-630 Spring 2008 Assignment 4 R. Pego

Problems due Wednesday April 30:

1. For $\boldsymbol{u} = (u, v)$, consider the system in the plane:

$$\boldsymbol{u}'(t) = f(\boldsymbol{u}) = \begin{pmatrix} u - v - u(u^2+v^2) \\ u + v - v(u^2+v^2) \end{pmatrix}$$

Note that this system has the explicit periodic solution $\boldsymbol{u}_*(t) = (u_*(t), v_*(t)) = (\sin t, -\cos t)$. If we linearize at the periodic solution, we get a linear system

$$\boldsymbol{u}'(t) = A(t)\boldsymbol{u}, \quad A(t) = Df(\boldsymbol{u}_*(t)),$$

with time-periodic coefficients, to which we can apply Floquet theory. Determine the Floquet exponents analytically!

For the next two problems, we suppose A is an $n \times n$ matrix and $\boldsymbol{g} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth and T-periodic in its first argument: $\boldsymbol{g}(t+T, \boldsymbol{u}) = \boldsymbol{g}(t, \boldsymbol{u})$ for all $t \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$. We seek to show (by applying the implicit function theorem in two different ways) that for small ε, the system

$$\boldsymbol{y}'(t) = A\boldsymbol{y} + \varepsilon \boldsymbol{g}(t, \boldsymbol{y}(t)) \tag{1}$$

has a small-amplitude T-periodic solution $\boldsymbol{y}(t+T) = \boldsymbol{y}(t)$, under an appropriate hypothesis on the eigenvalues of A. Part of the game is to figure out what the hypothesis should be.

2. (Method 1—snapshop map) Let the solution to the initial value problem for the system (1) with initial value

$$\boldsymbol{y}(0) = \boldsymbol{p} \in \mathbb{R}^n$$

be denoted $\varphi(t, \boldsymbol{p}, \varepsilon)$. A T-periodic solution of (1) corresponds to an initial condition $\boldsymbol{p}$ satisfying

$$F(\boldsymbol{p}, \varepsilon) := \varphi(T, \boldsymbol{p}, \varepsilon) - \boldsymbol{p} = 0.$$

What happens when $\varepsilon = 0$? Find a formula for the Jacobian matrix $(\partial F_i/\partial p_j(0,0))$ and describe a condition on the eigenvalues of A that ensure the Jacobian matrix is nonsingular. How can you infer that for all small ε a periodic solution of (1) exists?

3. (Method 2—function spaces) For $k = 0, 1$ let X_k be the vector space of T-periodic C^k functions $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^n$ with the norms

$$\|\boldsymbol{u}\|_{X_0} = \sup_{t \in [0,T]} |\boldsymbol{u}(t)|, \qquad \|\boldsymbol{u}\|_{X_1} = \|\boldsymbol{u}\|_{X^0} + \|\boldsymbol{u}'\|_{X^0}.$$

Define $F : X_1 \times \mathbb{R} \to X_0$ by

$$F(\boldsymbol{u}, \varepsilon) = \boldsymbol{u}' - A\boldsymbol{u} - \varepsilon \boldsymbol{g}(t, \boldsymbol{u})$$

(a) Note $\boldsymbol{u} \to L\boldsymbol{u} := F(\boldsymbol{u}, 0)$ is linear. Use the variation of parameters formula to prove that for each $\boldsymbol{w} \in X_0$, there is a unique T-periodic solution $\boldsymbol{v} \in X_1$ of $\boldsymbol{v}' = A\boldsymbol{v} + \boldsymbol{w}(t)$, under an appropriate condition on the eigenvalues of A.
(b) Infer that then $L = F_{\boldsymbol{u}}(0,0)$ has a bounded inverse $K = L^{-1} : X_0 \to X_1$. How can you infer that for all small ε a periodic solution of (1) exists?

4. Show that a smooth system of the form

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{x} + o(|\boldsymbol{x}|)$$

can be transformed to the normal form

$$\boldsymbol{x}' = \begin{pmatrix} x_1 + a x_1 x_2 \\ b x_2^2 \end{pmatrix} + o(|\boldsymbol{x}|^2).$$