Problems due Wednesday April 8:

1. Consider an autonomous system in $\mathbb{R}^2$ of the form

$$x'(t) = f(x,y), \quad y'(t) = 0,$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is globally $C^1$. Let $(T_-, T_+)$ denote the interval of maximal existence of the solution with given initial data $(x(0), y(0)) = (x_0, y_0)$. Regarding $T_+ \in (0, \infty]$ as a function of $y_0$, provide an example that shows that $T_+$ is not necessarily continuous. (Use the usual notion of infinite limits: neighborhoods of $\infty$ have the form $(M, \infty]$.)

2. Suppose $\phi : [0, \infty) \to \mathbb{R}$ is twice continuously differentiable, and

$$\phi''(t) + e^{-t}\phi(t) = 0$$

for all $t > 0$. If $\int_0^\infty |\phi(t)|^2 \, dt < \infty$, show that $\phi(t) \equiv 0$ on $[0, \infty)$.

3. For $y = (y_1, y_2)$, consider the planar system

$$y_1' = -y_2 + y_1(1 - r^2), \quad y_2' = y_1 + y_2(1 - r^2),$$

where $r^2 = y_1^2 + y_2^2$. Let $y(t, \tau, \xi)$ denote the solution which passes through $\xi \in \mathbb{R}^2$ at time $t = \tau$. Note that for $\xi = (1, 0)$, $y(t, \tau, \xi) = (\cos(t - \tau), \sin(t - \tau))$.

(a) Explicitly calculate the linear system of differential equations for which the Jacobian matrix

$$\frac{\partial y}{\partial \xi}(t, 0, (1, 0))$$

is a fundamental solution matrix.

(b) Use the given information to exhibit a periodic solution of period $2\pi$ for the linear system in part (a).

(c) Find the Floquet multipliers for the linear system in part (a).

4. Find the $\omega$-limit set of each solution (i.e., corresponding to every possible initial value) for the $2 \times 2$ system

$$x' = x(1 - r) - y(1 - r)^2 - \frac{y^3}{r^2}, \quad y' = y(1 - r) + x(1 - r)^2 + \frac{xy^2}{r^2},$$

where $r^2 = x^2 + y^2$. (At the origin, the quotients here are replaced by their limiting values 0.)
5. (Stability of the unstable manifold) In \( \mathbb{R}^k \times \mathbb{R}^l \) consider the system (N):

\[
\begin{align*}
x'(t) &= Sx + F(x, y), \\
y'(t) &= Uy + G(x, y),
\end{align*}
\]

where we assume as usual that for some \( a < 0 < b, K, \delta, \rho > 0 \),

(i) \( |e^{tS}| \leq Ke^{at} \) and \( |e^{-tU}| \leq Ke^{-bt} \) for all \( t \geq 0 \),

(ii) \( F \) and \( G \) are globally \( C^1 \) with \( F, G, DF \) and \( DG \) all zero at \((0, 0)\), and \( \|DF\|_0, \|DG\|_0 < \delta \).

(Recall \( \| \cdot \|_0 \) is the sup norm.)

Assuming \( \delta \) is small enough, system (N) has an unstable manifold of the form

\[ \hat{W}_\beta = \{ (x, y) \mid x = \beta(y) \}. \]

Suppose \( \beta : \mathbb{R}^l \to \mathbb{R}^k \) is \( C^1 \) with \( \beta(0) = 0 \) and assume \( \|D\beta\|_0 < \rho \).

(a) Show that for all \( \eta \in \mathbb{R}^l \),

\[
S\beta(\eta) + F(\beta(\eta), \eta) = D\beta(\eta)(U\eta + G(\beta(\eta), \eta)).
\]

(Consider the system satisfied by \( x(t) = \beta(y(t)) \) at \( t = 0 \) with appropriate initial data.)

(b) Given any solution \((x(t), y(t))\) of system (N), show \( u(t) = x(t) - \beta(y(t)) \) satisfies a system of the form

\[
u'(t) = Su(t) + H(u(t), y(t)),
\]

where \( |H(u, y)| \leq \delta(1 + \rho)|u| \) for all \((u, y) \in \mathbb{R}^k \times \mathbb{R}^l\),

(c) From the result of part (b), prove that \( |u(t)| \to 0 \) as \( t \to \infty \), provided \( a + K\delta(1 + \rho) < 0 \).