21-630 Spring 2008 Assignment 3 — due Wednesday March 26 R. Pego

1. Consider an autonomous system in $\mathbb{R}^2$ of the form

$$x'(t) = f(x,y), \quad y'(t) = 0,$$

where f is globally C^1. For the solution with $(x(0), y(0)) = (x_0, y_0)$, denote the right endpoint of the interval of maximal existence by $T_+(x_0, y_0) \in (0, \infty]$. Must this endpoint depend continuously on the data? Prove or give a counterexample. (Use the usual notion of infinite limits: neighborhoods of ∞ have the form $(M, \infty]$.)

2. (A Howland-type Floquet theory) Let $t \mapsto A(t)$ be a C^1 $n \times n$ matrix-valued function on $\mathbb{R}$ that is periodic with period T: $A(t+T) = A(t)$ for all t. Our objective is to establish a relation between the non-autonomous ODE system

$$\frac{d}{dt}\boldsymbol{y}(t) = A(t)\boldsymbol{y}, \tag{1}$$

where $\boldsymbol{y} : \mathbb{R} \to \mathbb{C}^n$, and the autonomous PDE system

$$\frac{\partial}{\partial t}\boldsymbol{u}(t,x) = A(x)\boldsymbol{u} - \frac{\partial}{\partial x}\boldsymbol{u} \tag{2}$$

where $t \mapsto \boldsymbol{u}(t, \cdot)$ maps $\mathbb{R}$ to the space X_T of C^1 functions $x \mapsto \boldsymbol{v}(x) \in \mathbb{C}^n$ that are T-periodic.
(a) Let $\Psi(t,s)$ be the state transition matrix for the system (1). Suppose $\mu \in \mathbb{C}$ is a Floquet multiplier for (1), so $\Psi(T,0)\boldsymbol{v} = \mu\boldsymbol{v}$ where $\boldsymbol{v} \in \mathbb{C}^n$. If $\lambda \in \mathbb{C}$ satisfies $e^{\lambda T} = \mu$, show that there exists $\boldsymbol{w} \in X_T$ such that $\boldsymbol{u}(t,x)$ satisfies (2), where

$$\boldsymbol{u}(t,x) = e^{\lambda t}\boldsymbol{w}(x).$$

(b) Given $\boldsymbol{u}_0 \in X_T$, let $\boldsymbol{w}(t,x) = \Psi(t+x, x)\boldsymbol{u}_0(x)$, so that

$$\frac{\partial}{\partial t}\boldsymbol{w}(t,x) = A(t+x)\boldsymbol{w}, \quad \boldsymbol{w}(0,x) = \boldsymbol{u}_0(x).$$

Find a formula for $\boldsymbol{u}(t,x)$ in terms of $\boldsymbol{w}$ so that $\boldsymbol{u}$ satisfies (2) and $\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x)$ for all x.

3. (Exterior algebra, linear systems, transport of area) Associated with the vector space $\mathbb{C}^n$ is the *exterior product space* $\Lambda^2\mathbb{C}^n$, spanned by objects of the form $v \wedge w$ for $v, w \in \mathbb{C}^n$. The map $(v, w) \mapsto v \wedge w$ is bilinear and anti-symmetric: For $a, b \in \mathbb{C}$, $u, v, w \in \mathbb{C}^n$,

$$(au + bv) \wedge w = a(u \wedge w) + b(v \wedge w), \quad v \wedge w = -w \wedge v.$$

The space $\Lambda^2\mathbb{C}^n$ can be represented as $\mathbb{C}^m$ with $m = \binom{n}{2} = n(n-1)/2$, where for the standard basis vectors $\boldsymbol{e}_j \in \mathbb{C}^n$, we make the identification $\boldsymbol{e}_j \wedge \boldsymbol{e}_k = \boldsymbol{e}_{j\wedge k} \in \mathbb{C}^m$ for $1 \le j < k \le n$, where $(j,k) \mapsto j \wedge k \in \{1,\dots,m\}$ is a one-to-one labeling.
(a) With the inner product $\langle v, w\rangle = \sum_j v_j w_j$ and 2-norm $\|v\| = \langle v, v\rangle^{1/2}$ on $\mathbb{R}^n$, prove the *Lagrange identity*

$$\|v\|^2\|w\|^2 - \langle v, w\rangle^2 = \|v \wedge w\|^2 = \sum_{i<j}(v_j w_k - v_k w_j)^2 = \frac{1}{2}\sum_{i,j=1}^{n}(v_j w_k - v_k w_j)^2.$$

(If angle θ between vectors v, w is defined by $\|v\|\|w\|\cos\theta = \langle v, w\rangle$, from the Lagrange identity it follows $\|v\|\|w\||\sin\theta| = \|v \wedge w\|$, so $v \wedge w$ can be interpreted as the oriented area of the parallelogram spanned by v and w.)
(b) An $n \times n$ matrix A induces a linear transformation $A_{(2)}$ of $\Lambda^2\mathbb{C}^n$ by using linearity to extend the rule

$$A_{(2)}(v \wedge w) = (Av) \wedge w + v \wedge (Aw) \qquad \text{for } v, w \in \mathbb{C}^n.$$

If $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ are two solutions of a linear system $\boldsymbol{y}'(t) = A(t)\boldsymbol{y}$, show that $\boldsymbol{y}_{(2)} = \boldsymbol{y}_1 \wedge \boldsymbol{y}_2$ satisfies $\boldsymbol{y}'_{(2)}(t) = A_{(2)}(t)\boldsymbol{y}_{(2)}$.
(c) Show that if λ_1 and λ_2 are eigenvalues of A, then $\lambda_1 + \lambda_2$ is an eigenvalue of $A_{(2)}$. If A is constant, describe an associated solution of $\boldsymbol{y}'_{(2)}(t) = A_{(2)}\boldsymbol{y}_{(2)}$ with time dependence proportional to $e^{(\lambda_1+\lambda_2)t}$.

4. (Poincaré recurrence theorem) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $\operatorname{div} f = 0$, so the flow of the system $y'(t) = f(y)$ preserves volumes. Denote the solution of the system with initial value $y(0) = y_0$ by $y(t) = \phi(t, y_0)$. Suppose that a bounded domain $\Omega \subset \mathbb{R}^n$ is invariant under the flow, so that $y_0 \in \Omega$ implies $\phi(t, y_0) \in \Omega$ for all $t \in \mathbb{R}$. Prove: If $y_0 \in \Omega$, then any neighborhood U of y_0 contains a point y_* such that there exist $t_k \to \infty$ as $k \to \infty$ with $\phi(t_k, y_*) \in U$. (Suggestion: First show that for any open ball $B \subset \Omega$, the sets $\phi(n, B)$, $n \ge 0$ integer, cannot be all pairwise disjoint, and deduce that for some $n > 0$, $B \cap \phi(n, B)$ is a nonempty open set. Obtain y_* as the intersection of a nested family of compact subsets of B.)