Problems due Monday Mar. 2:

1. Let $A, B : \mathbb{R} \to \mathbb{C}^{n \times n}$ be continuous, let $C \in \mathbb{C}^{n \times n}$ be given, and let $X(t)$ be the matrix-valued solution of the linear initial-value problem

$$X'(t) = A(t)X + XB(t), \quad X(0) = C.$$  

(a) Show that $X(t) = \Phi(t)C\Psi(t)$, where $\Phi(t)$ and $\Psi(t)$ are fundamental matrices for $y' = A(t)y$ and $z' = zB(t)$ respectively.

(b) If $A$ and $B$ are constant matrices whose eigenvalues all have negative real part, show that the unique solution of the matrix equation

$$AX + XB = C$$

is

$$Z = -\int_0^\infty e^{At}Ce^{Bt}dt.$$

2. (The Evans function and its derivative) Let $A : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}^{n \times n}$ be smooth, and let $\xi, \eta \in \mathbb{R}^{n \times 1}$ be given (column) vectors. Consider the following transmission problem: Find $\mu \in \mathbb{R}$ and $y : [t_0, t_1] \to \mathbb{R}^{n \times 1}$ such that

$$\partial_t y(t) = A(t, \mu)y(t), \quad t \in [t_0, t_1], \quad y(t_0) = \xi, \quad \eta^Ty(t_1) = 0.$$  

(Here $\eta^T$ denotes the transpose.) Let $\phi(t, \mu) = y(t)$ as determined by the initial condition at $t_0$, ignoring the “transmission condition” at $t_1$. Let $\psi(t, \mu) = z(t) \in \mathbb{R}^{1 \times n}$ as determined by the adjoint initial-value problem

$$\partial_t z(t) = -z(t)A(t, \mu), \quad t \in [t_0, t_1], \quad z(t_1) = \eta^T.$$  

(a) Show that the Evans function $E(\mu) := \psi(t, \mu)\phi(t, \mu)$ does not depend upon $t$. (Solutions of the transmission problem correspond to zeros of $E$. The Evans function is named after a postdoctoral researcher at NIH who studied the stability of the nerve impulse in the 1970s.)

(b) Prove the Melnikov-type formula

$$E'(\mu) = \int_{t_0}^{t_1} \psi(t, \mu)\partial_\mu A(t, \mu)\phi(t, \mu)dt.$$  

3. Prove that every Lyapunov exponent for a time-periodic linear ODE system is the real part of a Floquet exponent.
4. (Evolutionary semigroups and Floquet theory) Let \( t \mapsto A(t) \) be a bounded \( C^1 \) \( n \times n \) matrix-valued function on \( \mathbb{R} \). Our objective is to establish a relation between the non-autonomous ODE system

\[
\frac{d}{dt} y(t) = A(t) y, \quad y : \mathbb{R} \to \mathbb{C}^n, \quad (1)
\]

and the time-autonomous PDE system

\[
\frac{\partial}{\partial t} u(t, x) = A(x) u - \frac{\partial}{\partial x} u, \quad u : \mathbb{R}^2 \to \mathbb{C}^n. \quad (2)
\]

(a) Let \( \Psi(t, s) \) be the state transition matrix for the system (1). Given \( u_0 : \mathbb{R} \to \mathbb{C}^n \) continuous, let \( v(t, x) = \Psi(t + x, x) u_0(x) \), so that

\[
\frac{\partial}{\partial t} v(t, x) = A(t + x) v, \quad v(0, x) = u_0(x).
\]

Find \( u(t, x) \) explicitly in terms of \( v \) so that (2) holds and \( u(0, x) = u_0(x) \).

(b) Suppose \( A(t + T) = A(t) \) for all \( t \in \mathbb{R} \), and \( \mu \in \mathbb{C} \) is a Floquet multiplier for (1), so \( \Psi(T, 0) w = \mu w \) where \( w \in \mathbb{C}^n \). If \( \lambda \in \mathbb{C} \) satisfies \( e^{\lambda T} = \mu \), show that there exists a \( T \)-periodic function \( u_0 : \mathbb{R} \to \mathbb{C}^n \) such that \( u(t, x) \) satisfies (2), where

\[
u(t, x) = e^{\lambda t} u_0(x).
\]

5. (Poincaré recurrence theorem) Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^1 \) and \( \text{div} f = 0 \), so the flow of the system \( y'(t) = f(y) \) preserves volumes. Denote the solution of the system with initial value \( y(0) = y_0 \) by \( y(t) = \phi(t, y_0) \). Suppose that a bounded domain \( \Omega \subset \mathbb{R}^n \) is invariant under the flow, so that \( y_0 \in \Omega \) implies \( \phi(t, y_0) \in \Omega \) for all \( t \in \mathbb{R} \). Prove: If \( y_0 \in \Omega \), then any neighborhood \( U \) of \( y_0 \) contains a point \( y_* \) such that there exist \( t_k \to \infty \) as \( k \to \infty \) with \( \phi(t_k, y_*) \in U \). (Suggestion: First show that for any open ball \( B \subset \Omega \), the sets \( \phi(n, B) \), \( n \geq 0 \) integer, cannot be all pairwise disjoint. Deduce that for some \( n > 0 \), \( B \cap \phi(n, B) \) is a nonempty open set. Obtain \( y_* \) as the intersection of a nested family of compact subsets of \( B \).)