21-630 Spring 2008 Assignment 2 R. Pego

Problems due Monday March 3:

1. Let

$$A = \begin{pmatrix} -2 & 0 & 0 \\ -2 & 1 & 3 \\ -1 & 0 & -2 \end{pmatrix}$$

Find some fundamental matrix $\Phi(t)$, and determine e^{tA}.

2. Find a matrix-valued function $A(t)$ such that $\Phi(t) = \exp\left(\int_0^t A(s)\,ds\right)$ does not satisfy $\Phi'(t) = A(t)\Phi(t)$.

3. (Strang splitting) Let A and B be $n \times n$ matrices. Show that for some constant C,

$$\left| e^{t(A+B)} - e^{tA/2} e^{tB} e^{tA/2} \right| \le Ct^2, \quad |t| \le 1.$$

(Suggestion: Use a few terms from the Taylor series.)

4. (Isospectral flows and Lax pairs) Suppose that A is a square matrix and that $L(t)$ is a square-matrix solution of the matrix ODE

$$L'(t) = AL - LA. \tag{1}$$

Show that $L(t)$ similar to $L(0)$ for all t: $L(t) = V(t)L(0)V(t)^{-1}$. (Find $V(t)$.) Show that even if $A = A(t)$ is nonconstant (but continuous), again the same conclusion follows. The pair (L, A) is called a *Lax pair* if (1) holds. If (L, A) is a Lax pair, show that (e^L, A) is also a Lax pair. Any others?

5. Let $X = C([0,\infty))$ be the Banach space of bounded continuous functions $u : [0,\infty) \to \mathbb{R}$ with norm $\|u\| = \sup_{t\ge 0} |u(t)|$. Let $b > 1$ and let $v \in X$ and suppose v is C^1. For $u \in X$ let

$$T(u)(t) = v(t) + \int_t^\infty e^{b(t-s)} \sin(u(s))\,ds.$$

(i) Show T is a strict contraction on X, i.e., show that for some constant $K \in (0, 1)$,

$$\|T(u) - T(v)\| \le K\|u - v\| \quad \text{for all } u, v \in X.$$

By the contraction mapping principle (see Theorem 1.240 in Chicone, e.g.) it follows T has a unique fixed point in X: $u = T(u)$. (ii) Show that this unique fixed point is a bounded solution of a differential equation of the form $u'(t) = f(u(t)) + g(t)$. (iii) Show that for every $g \in X$ this equation has a unique solution $u \in X$. (E.g., show there is a solution in X, and any other solution is not in X.)