21-630 Spring 2008 Assignment 1 R. Pego

**Problems due Wednesday Feb. 6:**

**1.** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, $y\colon \mathbb{R} \to \mathbb{R}$ is $C^1$ and bounded, and $y'(t) = f(y(t))$ for all $t \in \mathbb{R}$. Suppose $y'(t_0) > 0$ for some $t_0$. Prove $y$ is increasing on $\mathbb{R}$ and $y_\infty = \lim_{t\to\infty} y(t)$ exists with $f(y_\infty) = 0$.

**2.** Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition in $y$ with $t$-dependent Lipschitz constant $K = K(t)$, so that

$$|f(t,y) - f(t,\hat{y})| \le K(t)|y - \hat{y}| \quad \text{for all } t, y, \hat{y}.$$

Fix $t_0$ and suppose $\phi_1$ and $\phi_2$ are continuous functions on $\mathbb{R}$ satisfying

$$\phi_j(t) = \phi_j(t_0) + \int_{t_0}^t f(s, \phi_j(s))\,ds + \varepsilon_j(t), \quad j = 1, 2.$$

Suppose $t \mapsto K(t)$ is continuous, and suppose $|\phi_1(t_0) - \phi_2(t_0)| \le \delta$, and $\varepsilon(t) = |\varepsilon_1(t) - \varepsilon_2(t)|$. Prove that for all $t \ge t_0$,

$$|\phi_1(t) - \phi_2(t)| \le \delta \exp \int_{t_0}^t K(s)\,ds + \varepsilon(t) + \int_{t_0}^t \varepsilon(s) K(s) \exp\left(\int_s^t K(r)\,dr\right) ds.$$

**3.** Consider the IVP for an $n$-component system

$$\boldsymbol{y}' = A(t)\boldsymbol{y} + \boldsymbol{b}(t)\ , \qquad \boldsymbol{y}(t_0) = \eta$$

where the $n \times n$ matrix $A$ and the vector-valued function $\boldsymbol{b}$ are continuous on an interval $I$ of the form $I = [t_0, t_1]$. Prove uniqueness, i.e., prove that there is *at most one* solution of the IVP on $I$. Use the strategy: compare two solutions, obtain an integral inequality for the 1-norm

$$|\boldsymbol{y}(t)| = \sum_{j=1}^n |y_j(t)|,$$

and use the Gronwall inequality.

**4.** Under the same conditions as in the previous problem, prove that *Picard's iteration method* converges to a solution of the IVP, as follows: We let $\boldsymbol{y}_0(t) = \eta$ for all $t \in I$, and inductively construct $\boldsymbol{y}_n(t)$ for $n = 1, 2, \ldots$ from

$$\boldsymbol{y}_n(t) = \eta + \int_{t_0}^t A(s)\boldsymbol{y}_{n-1}(s) + \boldsymbol{b}(s)\,ds.$$

Show (i) that $r_n(t) = |\boldsymbol{y}_n(t) - \boldsymbol{y}_{n-1}(t)|$ satisfies inequalities of the form $r_1(t) \le M(t - t_0)$, $r_n(t) \le \int_{t_0}^t K r_{n-1}(s)\,ds$ for $n \ge 2$, and infer

$$r_n(t) \le M K^{n-1} \frac{(t - t_0)^n}{n!} \quad \text{for all } t \in I.$$

Then it remains to prove
(ii) $\boldsymbol{y}(t) = \eta + \sum_{n=1}^\infty (\boldsymbol{y}_n(t) - \boldsymbol{y}_{n-1}(t))$ exists and is continuous, and
(iii) $\boldsymbol{y}(t)$ satisfies the IVP (integral version first).

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**5.** Let $\alpha : [0,T] \to \mathbb{R}$ be an increasing function (not necessarily continuous) with $0 = \alpha(0) = \alpha(0^+)$ and let $f : [0,T] \to \mathbb{R}$ be continuous. For $0 \le t \le T$ and any partition $P = \{t_0, \ldots, t_n\}$ of $[0,t]$, with

$$0 = t_0 < t_1 < \ldots < t_n = t, \qquad \|P\| = \max_j |t_j - t_{j-1}|,$$

note that the following Stieltjes integral exists:

$$\int_0^t f(s)\,d\alpha(s) = \lim_{\|P\|\to 0} \sum_{j=1}^n f(t_j)(\alpha(t_j) - \alpha(t_{j-1})).$$

Suppose $f$ satisfies

$$0 \le f(t) \le \delta + \int_0^t f(s)\,d\alpha(s), \qquad 0 \le t \le T.$$

where $\delta \ge 0$. Prove that $f(t) \le \delta e^{\alpha(t)}$ for $0 \le t \le T$.
(Suggestion: 'Discretize' the proof of Gronwall's inequality, using a modulus of continuity for $f$.)