Homework Assignment 8


REMINDER: The second midterm test will be a take-home test. The test will be handed out in class Wednesday April 8, and is due at the beginning of class Friday, April 10. The syllabus for the test consists of the topics covered in class or on the homework up through Friday April 3, as in the class schedule/summary.

1. Suppose $U$ is an $n \times n$ matrix whose columns are orthonormal. Prove $\det U \in \{-1, 1\}$.

2. Prove that the determinant of a lower triangular matrix is the product of the diagonal entries. That is, if $A = (a_{ij})$ is an $n \times n$ matrix and $a_{ij} = 0$ whenever $i < j$, then show that

$$\det A \stackrel{\text{def}}{=} \prod_{\sigma \in S_n} \text{sgn } \sigma \prod_{j=1}^{n} a_{\sigma(j)d} = \prod_{j=1}^{n} a_{jj}.$$ 

3. The aim of this problem is to show how the determinant of a linear transformation $T$ on a finite-dimensional vector space $V$ can be defined in an unambiguous way. Suppose $A = (v_1, \ldots, v_n)$ and $B = (w_1, \ldots, w_n)$ are two ordered bases for $V$. Then with respect to each basis, $T : V \to V$ is represented by matrices

$$A = M_{AA}(T), \quad B = M_{BB}(T).$$

Prove that $\det A = \det B$. This means that any matrix representing $T$ in any basis has the same determinant, and we can define $\det T$ unambiguously to be this common value.

4. Use column reorderings and shears to evaluate the determinant of the following companion matrix, and show that the $n \times n$ determinant

$$\begin{vmatrix} x & -1 & 0 & \ldots & 0 \\ 0 & x & -1 & 0 & \ldots \\ 0 & 0 & x & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_0 & c_1 & c_2 & \ldots & (c_{n-1} + x) \end{vmatrix} = \pm \begin{vmatrix} -1 & 0 & 0 & \ldots & x \\ x & -1 & 0 & \ldots & 0 \\ 0 & x & -1 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_1 & c_2 & \ldots & (c_{n-1} + x) & c_0 \end{vmatrix} = \pm p(x),$$

where

$$p(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1} + x^n.$$ 

Which signs above are correct? (The second matrix is obtained from the first by moving the first column to the last; every other column moves one place to the left. The original matrix has the form $xI + A$ where $a_{ij} = -1$ if $j = i + 1$ and $a_{nj} = c_{j-1}$, $j = 1, \ldots, n$.)

5. (Arbitrary inner products and the SVD) Suppose that $(\cdot, \cdot)_a$ is an inner product on $\mathbb{R}^n$. (We use the subscript $a$ to distinguish this from the standard inner product $\langle v, w \rangle = v^T w$.)

(a) Show there is a symmetric, invertible matrix $\mathcal{A}$, such that for any $v, w \in \mathbb{R}^n$ (column vectors),

$$(v, w)_a = v^T A w.$$ 

(b) Let $A = UDV^T = \sum_{j=1}^{n} u_j \kappa_j v_j^T$ be the singular value decomposition of $A$. Prove that

$$\kappa_1 = u_1^T A v_1 \leq \sqrt{u_1^T A u_1} \sqrt{v_1^T A v_1} \leq \kappa_1.$$ 

Then prove that $A v_1 = \kappa_1 v_1$, hence $u_1 = v_1$.

(c) Develop an induction argument to deduce $u_k = v_k$ for all $k$. That is, prove $\mathcal{U} = \mathcal{V}$.

(d) Deduce that, in terms of the basis $u_1, \ldots, u_n$ coming from the columns of $\mathcal{U}$, the inner product can be written

$$(v, w)_a = \sum_{m=1}^{n} \kappa_m b_m c_m, \quad \text{where} \quad v = \sum_{i=1}^{n} b_i u_i, \quad w = \sum_{j=1}^{n} c_j u_j.$$