Homework Assignment 6
Assigned Fri 2/20. Due Fri 2/27. 21-341 Spring 2015 (R. Pego)

REMINIDER: The first midterm test will be held Monday March 2, in class. The test is closed book, no notes or aids permitted. The syllabus for the test consists of the topics covered in class or on the homework up through Monday Feb. 23, see the class schedule/summary.

1. Prove that
\[
\left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} a_j^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right)
\]
for all real numbers \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\). [The fact that \(\frac{1}{2} = 1\) is sort of relevant, as you may suspect!]

2. Let \(V\) be a finite-dimensional vector space with inner product, and let \(W\) be a subspace of \(V\). Define \(W^\perp\) as the subspace of vectors \(v \in V\) for which the inner product \((v, w) = 0\) for all \(w \in W\). (I.e., \(v \perp w\) for all \(w \in W\)). Prove that \(W\) and \(W^\perp\) are complementary subspaces in \(V\), whence \(\dim W + \dim W^\perp = \dim V\).

3. Given two vectors \(v\) and \(w\) in \(\mathbb{R}^n\), recall from problem 4.4 the definition of the wedge product \(v \wedge w\), whose components are indexed by pairs \((i, j)\) with \(1 \leq i < j \leq n\). The norm of \(v \wedge w\) is its Euclidean norm as a vector in \(\mathbb{R}^{m_n}\) where \(m_n = \binom{n}{2}\): Thus
\[
\|v \wedge w\|^2 = \sum_{1 \leq i < j \leq n} |v_i w_j - v_j w_i|^2.
\]

(a) Prove the following sharp improvement of the Cauchy-Schwarz inequality:
\[
\|v\|^2 \|w\|^2 = (v, w)^2 + \|v \wedge w\|^2
\]
(Hint: What happens to the sum for \(\|v \wedge w\|^2\) if you include terms with \(i = j\)? All \(i \geq j\)?)

(b) Infer from the previous part that for any linear map \(T: \mathbb{R}^n \to \mathbb{R}^n\) that is isometric (i.e., “orthogonal”) we have
\[
\|(Tv) \wedge (Tw)\| = \|v \wedge w\|.
\]

4. Suppose \(n \geq 2\), and \(v, w\) are nonzero vectors in \(\mathbb{R}^n\). Prove that there is a linear isometry (i.e., an “orthogonal” map) \(T: \mathbb{R}^n \to \mathbb{R}^n\) such that \(Tv = \|v\| e_1\) and \(Tw \in \text{span}\{e_1, e_2\}\), where \(e_j\) denotes the \(j\)th standard unit basis vector.

5. A Pythagorean theorem for areas. Suppose \(n \geq 2\), and \(v, w\) are nonzero vectors in \(\mathbb{R}^n\).

(a) With \(T\) as in the previous question, let \(\hat{v} = Tv = \|v\| e_1\), \(\hat{w} = Tw\). Let \(\hat{P}\) be the parallelogram whose corners are at the points \(0, \hat{v}, \hat{w}, \hat{v} + \hat{w}\). Let \(|\hat{P}|\) denote the area of \(\hat{P}\). Show \(|\hat{P}| = \|\hat{v} \wedge \hat{w}\|\).

(b) Whenever \(1 \leq i < j \leq n\), let \(P_{ij}\) denote the parallelogram in \(\mathbb{R}^2\) whose corners are the points
\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} v_i \\ w_j \end{pmatrix}, \begin{pmatrix} w_i \\ v_j \end{pmatrix}, \begin{pmatrix} v_i + w_i \\ v_j + w_j \end{pmatrix}.
\]
Show that this parallelogram has area \(|P_{ij}| = |v_i w_j - v_j w_i|\).

(c) Deduce that if \(P\) is the parallelogram in \(\mathbb{R}^n\) whose corners are the points \(0, v, w, v + w\), then the area of \(P\) satisfies
\[
|P|^2 = \sum_{1 \leq i < j \leq n} |P_{ij}|^2.
\]
(You may assume \(|P| = |\hat{P}|\).)