1. Suppose \( Y \) is a subspace of some vector space \( V \) over a field \( F \). If \( V \) is finitely generated, prove that \( Y \) is finitely generated. [This was stated in class as Proposition S1 but not proved. Don’t use any results that were proved in class afterward!]

2. Let \( X, Y, Z \) be subspaces of a vector space \( V \) over a field \( F \). Then of course
\[
(X \cap Y) \cup (X \cap Z) = X \cap (Y \cup Z)
\]
because this is true for any sets.

(a) Show that it is not necessarily true, however, that
\[
(X \cap Y) + (X \cap Z) = X \cap (Y + Z).
\]

(b) Show that necessarily, instead,
\[
(X \cap Y) + (X \cap Z) = X \cap (Y + (X \cap Z)).
\]

3. Let \( B = \{u_1, \ldots, u_m\} \) be a set of \( m \) linearly independent vectors in a vector space \( V \), and let \( Y = \text{span} B \). Supposing \( w \notin Y \), let \( C = B + w = \{u_1 + w, \ldots, u_m + w\} \), and let \( Z = \text{span} C \). Show that \( C \) is a linearly independent set, and determine \( \dim Y \cap Z \).

4. Suppose \( \dim(V) = 1 \), and \( T \in \mathcal{L}(V, V) \). Show that \( \exists \alpha \in F \) such that \( T(v) = \alpha v \) for all \( v \in V \).

5. Use the differentiation map on the vector space \( P_n(\mathbb{R}) \) to show that the range and the null space (kernel) of a linear transformation \( T: V \to V \) may have nontrivial intersection. What are the dimensions of the range and the kernel in this case? [As previously, \( P_n(F) \) is the vector space of polynomial functions \( f: F \to F \) with degree \( \deg f \leq n \).]

Suppose one only wants to detect errors while transmitting a message. The usual way to do this is to transmit a block of your message (say 7 bits), and then transmit one ‘parity bit’. This parity bit is chosen so that the total number of 1’s in your message block (including the parity bit) is even. In our setup from class, this amounts to the following.

6. Let \( F = \{0, 1\}, \ n \geq 2, \ V = F^n \) and define \( C = \text{span}\{e_1 + e_n, e_2 + e_n, \ldots, e_{n-1} + e_n\} \). Show that \( \dim(C) = n - 1 \), and for all \( i \in \{1, \ldots, n\}, \ e_i \notin C \).

7. (This one is optional: Try it, but don’t turn it in.) Let \( F = \{0, 1\} \). Find \( n \), and a subspace \( C \subset F^n \) such that \( \dim(C) > \frac{n}{2} \), and for all \( i, \ e_i \notin C \), and for all \( i \neq j, \ e_i + e_j \notin C \). [This shows it is possible to transmit a message, and correct for at most one error, by transmitting less than twice your message.]