1. Determine whether \((2, 2, 2)\) belongs to the subspace of \(\mathbb{R}^3\) generated by \((1, 3, 4), (1, -1, -1)\) and \((3, 1, 2)\). Explain your reasoning.

2. Let \(V\) be a vector space over a field \(F\), and suppose \(B = (u_1, \ldots, u_n)\) is a list of \(n\) distinct nonzero vectors in \(V\). Define a new list by
   \[
   \hat{B} = \{u_1, u_2 - u_1, \ldots, u_n - u_{n-1}\}
   \]
   obtained by subtracting each vector \(u_j\) from its right-hand neighbor \(u_{j+1}\), for \(j = 1, \ldots, n-1\).
   
   (a) Show that if the list \(B\) spans \(V\) then so does the list \(\hat{B}\).
   
   (b) Show that if \(B\) is linearly independent then so is \(\hat{B}\).

3. Let \(F = \mathbb{R}\) or \(F = \mathbb{C}\), and let \(P_n(F)\) be the set of polynomial functions \(f : F \to F\) with degree \(\deg f \leq n\). Suppose \(f_0, f_1, \ldots, f_n \in P_n(F)\) all vanish at \(x = 1\): \(f_j(1) = 0\) for all \(j\). Show that \(\{f_0, f_1, \ldots, f_n\}\) cannot be linearly independent in the vector space \(P_n(F)\).

4. So far we’ve dealt with the spans of finite non-empty sets. Let’s deal now with infinite sets. Recall that when \(B\) is a finite subset of \(V\), then \(\text{span}(B)\) is the minimal subspace \(U \subset V\) that contains \(B\). We want to use this property to define \(\text{span}(B)\) when \(B\) is infinite. Of course, before we can make such a definition, we must show that the “minimal subspace of \(V\) containing \(B\)” exists, and is unique! So let \(B \subset V\) be any subset (not necessarily finite).
   
   (a) Let \(U\) the intersection of all subspaces of \(V\) that contain \(B\). Show that \(U\) is a minimal subspace of \(V\) that contains \(B\). [This shows existence.]
   
   (b) If \(U\) and \(U'\) are two minimal subspaces of \(V\) that contain \(B\), then show \(U = U'\). [This shows uniqueness. Now, it is legitimate for us to define \(\text{span}(B)\) to be the minimal subspace of \(V\) containing \(B\).]
   
   (c) Using the definition of span above, what is \(\text{span}(\emptyset)\)?
   
   (d) If span is defined as above, show that \(\text{span}(B) = \{\sum_{i=1}^n \alpha_i v_i \mid n \in \mathbb{N}, \alpha_i \in F, v_i \in B\}\) if \(B \neq \emptyset\). [That is, \(\text{span}(B)\) is the set of all finite linear combinations of elements in \(B\).]

5. Let \(F = \mathbb{Z}_2\), the field with only two elements. Suppose \(A\) is a set of \(k\) distinct prime numbers. Let us say that a natural number \(n > 0\) is \(A\)-philic if its factorization into primes (not necessarily distinct) contains only elements of \(A\). Prove that if \(B = \{n_1, \ldots, n_{k+1}\}\) is a set of \(k + 1\) distinct \(A\)-philic numbers, then there is a subset \(C \subset B\) such that the product of the numbers in \(C\) is a perfect square. [Hint: Associate to each \(n_j\) a vector in \(F^k\), and use facts about linear independence in \(F^k\).]