Homework Assignment 2
Assigned Fri 1/23. Due Fri 1/30.

1. The conclusion of both subparts below will follow directly from a general theorem we will prove later. However it’s worth while doing them out explicitly by hand at least once!
   (a) Suppose \( u_1, u_2 \) are any two linearly independent vectors in \( \mathbb{R}^2 \), then show (by direct computation) that \( \text{span}\{u_1, u_2\} = \mathbb{R}^2 \).
   (b) Let \( V = \mathbb{R}^3 \), and \( U \subseteq \mathbb{R}^3 \) be the plane \( x_1 + x_2 + x_3 = 0 \). Show (by direct computation) that if \( u_1, u_2 \) are any two linearly independent vectors in \( U \), then \( U = \text{span}\{u_1, u_2\} \).

2. Let \( V \) be a vector space over a field \( F \).
   (a) Suppose \( U \) and \( W \) are two subspaces of \( V \). Is \( U \cup W \) always a vector space? If yes, prove it. If no, furnish a counter example.
   (b) Same question as in part (a) for \( U \cap W \).
   (c) Define \( U + W = \{u + w \mid u \in U, w \in W\} \). If \( U,W \) are subspaces of \( V \), then show that \( U + W \) is also a subspace of \( V \).
   (d) If \( V = \mathbb{R}^3 \), \( F = \mathbb{R} \), \( U = \{\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) \mid x_1 + x_2 = 0\} \), and \( W = \{\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) \mid x_1 + x_2 + x_3 = 0\} \), then compute \( U + W \).

3. Let \( V \) be a vector space over a field \( F \). Recall that a list of vectors \( u_1, \ldots, u_n \) in \( V \) is called a basis for \( V \) if every element \( u \) in \( V \) has a unique representation as a linear combination
   \[
   u = \alpha_1 u_1 + \ldots + \alpha_n u_n \quad \text{with} \quad (\alpha_1, \ldots, \alpha_n) \in F^n.
   \]
   Prove that \( u_1, \ldots, u_n \) is a basis if and only if it is a minimal generating list, meaning that it generates \( V \) but no smaller sublist generates \( V \).

4. Let \( F = \mathbb{Z}_2 \) and let \( V = F^2 \). How many elements are in \( V \)? How many subspaces? How many different bases does \( V \) have?

5. Let \( V \) and \( W \) be vector spaces over a field \( F \), and let \( Z \) be the set of all vector space homomorphisms \( T: W \to V \). Recall this means that for all \( a, b \in W \) and all \( \gamma \in F \),
   \[
   (i) \quad T(a + b) = T(a) + T(b) \quad \text{and} \quad (ii) \quad T(\gamma a) = \gamma T(a).
   \]
   With addition and scalar multiplication on \( Z \) defined as for arbitrary functions from \( W \) to \( V \), prove that \( Z \) is a vector space.