Optional Homework Assignment 10
Assigned Fri 4/24. Due Fri 5/1. 21-341 Spring 2015 (R. Pego)

NOTE 1: This assignment is optional — it is recommended, but not required. It may be turned in to improve your homework score or for practice with the material, as you wish.

NOTE 2: The final examination will be held Thursday, May 7, from 8:30-11:30am, in room 2122 of Doherty Hall. The syllabus for the test consists of the topics covered in class or on the homework up through the last day of class, Friday May 1.

For problems 1-3 below, let $V$ be a vector space of dimension $n$ over an arbitrary field $F$, and let $V^*$ be the dual space of $V$.

1. Let $V$ have basis $v_1, \ldots, v_n$, and let $f_1, \ldots, f_n$ be the dual basis of $V^*$. Let $Z = \text{span}\{v_1, \ldots, v_k\}$, $k < n$. Show that $Z^\perp = \text{span}\{f_{k+1}, \ldots, f_n\}$, and infer that $\dim Z + \dim Z^\perp = \dim V$.

2. If $S, T \in \mathcal{L}(V, V)$, which is always true: $(ST)^* = T^* S^*$, or $(ST)^* = S^* T^*$? Prove your answer.

3. If $T \in \mathcal{L}(V, V)$ and and $p(T) = 0$ for some polynomial $p \in F[x]$, prove that $p(T^*) = 0$.

4. An $n \times n$ matrix $A = (a_{ij})$ is called a graph Laplacian if it is symmetric, $a_{ij} \in \{0, -1\}$ whenever $i \neq j$, and $a_{ii} = -\sum_{j \neq i} a_{ij}$.
   The corresponding graph with nodes labeled $1, \ldots, n$ has an edge connecting $i$ and $j$ exactly when $a_{ij} = -1$. Here is a simple example of a labeled graph and its Laplacian matrix:

   ![Graph Diagram](image)

   Suppose that $A$ is the graph Laplacian for a connected graph (meaning that every two nodes are connected by some sequence of edges). Show that if $b = (b_i) \in \mathbb{R}^n$ is given, then there exists a solution $x = (x_j) \in \mathbb{R}^n$ to the linear system $Ax = b$ if and only if $\sum_{i=1}^n b_i = 0$. [Hint: if $Ax = 0$, look at the node where $x_j$ is maximized, and consider the neighbors.]

5. Suppose $V$ is a vector space over $\mathbb{C}$ of dimension $n$. Let $T \in \mathcal{L}(V, V)$.

   (a) Let $v_1, \ldots, v_k$ of $T$ be a Jordan chain for some eigenvalue $\alpha \in \mathbb{C}$. Let $v_1, \ldots, v_n$ be a basis containing the Jordan chain, and let $f_1, \ldots, f_n$ be the dual basis of $V^*$. Prove that $f_1, \ldots, f_k$ (but not in that order) are elements of a Jordan chain of $T^*$ for the same $\alpha$.

   (b) Using the previous part, deduce that $T^*$ has Jordan canonical form the same as that of $T$. In particular, if $J = \mathcal{M}_{AA}(T)$ is the Jordan canonical form of $T$ with respect to a basis $A$ of $V$ consisting of Jordan chains of $T$, describe a basis $A^*$ of $V^*$ consisting of Jordan chains of $T^*$ such that $J = \mathcal{M}_{A^*A^*}(T^*)$. 