

## 21-236 Math Studies: Problem Seminar Batch 4

Let's try mixing up new groups:

I: Cowan, Durback, Gates, Xia

J: Butler, Cham, Yu

K: Condon, Oliver, Schaeffer

L: Cameron, Wolf, Ziegler

The groups should start working on problems according to (I,J,K,L)=(13,14,15,16), and continue freely as time allows.

13. Two opponents take turns on an infinite square grid: A (the aggressor), and D (the defender). On his turn, A can advance one square east, northeast, or southeast, if the target square is not blocked. On his turn, D can block any one square on the board, permanently. Show that D can ultimately block A completely.
14. The function  $f: (0, \infty) \rightarrow (0, \infty)$  given by  $f(x) = xe^x$  is bijective. If  $xe^x = t$ , show  $x = f^{-1}(t)$  satisfies:
  - (i)  $x = \log t + O(\log \log t)$  as  $t \rightarrow \infty$ ,
  - (ii)  $x = \log t - \log \log t + O(\log \log t / \log t)$  as  $t \rightarrow \infty$ .(Recall the meaning of "big-oh notation": We say  $f(t) = O(g(t))$  as  $x \rightarrow \infty$  if there exist positive constants  $b, c$  such that  $|f(t)| \leq c|g(t)|$  whenever  $t > b$ .)
15. a) Let  $V = \{x_1, \dots, x_n\}$  be a set partially ordered by a relation  $\leq$  (this means that the relation is reflexive, antisymmetric and transitive). Call an  $n \times n$  matrix  $(a_{ij})$  compatible if

$$a_{ij} \neq 0 \rightarrow x_i \leq x_j.$$

Show that the sum, product, and (if it exists) the inverse of compatible matrices is compatible.

- b) Prove that there exists a unique function  $\mu$  defined on  $V \times V$  such that
  - (i)  $\mu(x, y) = 0$  if  $x \not\leq y$
  - (ii)  $\mu(x, x) = 1$
  - (iii)  $\sum_{x \leq y \leq z} \mu(x, y) = 0$  ( $x < z$ ).
16. (a) A sequence of partitions of the  $n$ -element set  $S$  is constructed as follows: We start with  $S$  itself and then we obtain the  $(i + 1)$ st partition from the  $i$ th by splitting any one of its classes having more than one element into two nonempty subsets. Thus the partition into one-element classes will be reached after  $n - 1$  steps. What is the number of ways of carrying out this procedure?
  - (b) Modify (a) by saying that the the  $(i + 1)$ st partition is obtained from the  $i$ th by splitting all classes having more than one element into two nonempty subsets. What is the number of ways of carrying out the procedure now?