

## 21-236 Math Studies: Problem Seminar Batch 3

The same groups should start working on problems according to  $(Z,Y,X,W)=(9, 10, 11, 12)$ , and continue freely as time allows.

9. Motivated by the ‘divide-and-conquer’ recursive approach to the problem 3(b), we seek an upper bound on the amount of work  $W(n)$  (number of operations) involved in pairing  $n$  electrons with  $n$  protons in  $\mathbb{R}^2$  such that no two linking segments intersect. By what was shown in seminar, this amount of work is bounded by a constant times  $n$ , plus the amount of work in pairing two subsets, with  $j$  and  $n - j$  pairs each, for some  $j$  satisfying  $0 \leq j \leq n - 1$ . That is, for some constant  $A$  we have

$$W(n) \leq An + W(j) + W(n - j - 1) \quad \text{for some } j \in \{0, \dots, n - 1\}.$$

Show that for an appropriate constant  $B$ , we have  $W(n) \leq Bn^2$  for all  $n \in \mathbb{N}$ . Identify the smallest  $B$  for which your proof works.

10. Show that any curve of length 1 in the plane can be covered by a rectangle of area  $1/4$ .
11. (a) Can a  $10 \times 10$  square be broken up into dominoes of size  $4 \times 1$ ?
- (b) Let  $b$  be a positive integer. For which pairs  $(c, d)$  can a  $c \times d$  square be broken up into dominoes of size  $1 \times b$ ?
- (c) Let  $a$  and  $b$  be positive integers. For which pairs  $(c, d)$  can a  $c \times d$  square be broken up into dominoes of size  $a \times b$ ?
12. Chaz chooses two integers between 2 and 99 inclusive, and he tells Pam their product and Sam their sum. The following dialog ensues:

Pam: “I don’t know the two numbers.”

Sam: “I already knew that.”

Pam: “Then now I know the two numbers.”

Sam: “Then now I know them, too.”

What are the two numbers? Can you prove your solution unique?