21-236 Math Studies: Problem Seminar Batch 2

The same groups should start working on problems according to (W,X,Y,Z)=(5, 6, 7, 8), and continue freely as time allows.

5. Find all functions $f : \mathbb{R} \to (0, \infty)$ such that $f(x + y) = f(x)f(y)$ and f is bounded on $[0, 1]$.

6. Let F_n denote the nth Fibonacci number ($F_{n+1} = F_n + F_{n-1}$, $n \geq 1$, $F_1 = F_0 = 1$). Show that for all n,
$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = \pm 1,$$
and indeed, all pairs of positive integers (x, y) that satisfy $x^2 - xy - y^2 = \pm 1$ and $x > y$ have the form (F_{n+1}, F_n) for some n. Deduce that the set of positive values of the polynomial $x(2-(x^2-xy-y^2)^2)$ on $\mathbb{N} \times \mathbb{N}$ is exactly the set of Fibonacci numbers.

7. Can you design *two loaded dice* so that rolling the pair gives the numbers $2, 3, \ldots 12$ with equal probability?

8. Can a cube be subdivided into a finite number of disjoint subcubes all of different size?