21-236 Math Studies: Problem Seminar

The plan for Spring semester is for problem seminar work to be done in small groups, which prepare and present solutions to problems. Groups will later work on a more substantial project and presentation, on a topic to be determined/negotiated by late February. Groups for the first week or so are:

X: Butler, Cameron, Schafer

Y: Cham, Condon, Durback, Xia

Z: Cowan, Oliva, Ziegler

W: Gast, Wolf, Yu

Groups should start work on problems according to (X,Y,Z,W)=(1,2,3,4), and continue freely if time allows.

1. Suppose we have a group of 8 people who wish to form clubs with the following properties:

   (a) Every club has exactly 4 people

   (b) Every set of three people lies in exactly one club

   (c) Every two clubs have exactly 0 or 2 people in common.

   Can this be done?

2. Let $[a, b]$ and $[c, d]$ be two closed bounded intervals in $\mathbb{R}$, and let S be the set of increasing functions $f : [a, b] \to [c, d]$. (Recall f is said to be *increasing* if $x_1 < x_2 \implies f(x_1) \leq f(x_2)$.) Show that for any sequence $(f_n)_{n\in\mathbb{N}}$ in S, there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a function $f \in S$ such that

$$f_{n_k}(x) \to f(x) \qquad \text{as } k \to \infty, \quad \textit{for every } x \in [a, b].$$

3. Suppose we are given two sets S and T in $\mathbb{R}^m$ each consisting of n distinct points ("protons" and "electrons").

   (a) Suppose we match up each proton with exactly one electron, and assign the pair an energy corresponding to the squared distance separating them. (This is the potential energy of a linear spring stretched by this distance.) For the two sets of n points, the minimum total energy we get defines the quantity

$$E(S, T) = \min_{\sigma\in B(S,T)} \sum_{(p,q)\in\sigma} |p - q|^2.$$

   Here $B(S, T)$ is the set of bijections $\sigma : S \to T$. Recall that such a function is properly identified with its graph, a subset of the Cartesian product: $\sigma \subset S \times T$. Now, define a 'distance' between S and T via

$$d(S, T) = \sqrt{E(S, T)}.$$

   Show that indeed d is a distance, i.e., it is a metric on the space of such sets.

   (b) Suppose $m = 2$ and no three of the $2n$ points are collinear. Show that there is a way to pair each proton with exactly one electron so that none of the n line segments connecting corresponding electrons and protons intersect.

4. Two people take turns cutting up a rectangular chocolate bar which is 6×8 squares in size. You are allowed to cut the bar only along a division between the squares and your cut can be only a straight line. The last player who can break the chocolate wins (and gets to eat the chocolate bar). Is there a winning strategy for the first or second player? What about the general case (the starting bar is $m \times n$)? (Apparently, this problem has befuddled a good number of serious research mathematicians.)