Proof that e is transcendental

To prove e is transcendental we must show it is not algebraic, i.e., there is no polynomial Q with rational coefficients $a_0, a_1, \ldots, a_n$ such that

$$Q(e) = a_n e^n + \ldots + a_1 e + a_0 = 0. \tag{1}$$

We basically follow an argument in *Conjecture and Proof* by M. Laczkovich. Suppose there is such a polynomial. We may assume $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $a_0 \neq 0$.

1. Let f be a polynomial of degree m. Then (repeated) integrations by parts gives

$$\begin{aligned} \int_0^k f(x)e^{-x}\,dx &= -f(x)e^{-x}\Big|_0^k + \int_0^k f'(x)e^{-x}\,dx \\ &= -(f(x) + f'(x) + \ldots + f^{(m)}(x) + 0)e^{-x}\Big|_0^k. \end{aligned}$$

Multiply by $a_k e^k$ and add up: Then

$$\sum_{k=0}^{n} a_k e^k \int_0^k f(x)e^{-x}\,dx = -\sum_{k=0}^{n} a_k (f(k) + f'(k) + \ldots + f^{(m)}(k)) \tag{2}$$

since $Q(e) = 0$. Our goal is to obtain a contradiction by showing there is an f so that (i) the LHS is small, and (ii) the RHS is a nonzero integer.

2. We will choose f of the form

$$f(x) = \frac{1}{(N-1)!} x^{N-1}(x-1)^N(x-2)^N \cdots (x-n)^N.$$

where N is a sufficiently large prime number. Note that for $0 \leq x \leq n$ we have

$$|f(x)| \leq \frac{n^{(n+1)N}}{(N-1)!} = \frac{A^N}{(N-1)!}, \quad A = n^{n+1}.$$

Hence the LHS of (2) is bounded by

$$|\mathrm{LHS}| \leq (n+1) \max_k |a_k|\, e^n \cdot n \cdot \frac{A^N}{(N-1)!} = \frac{C_n A^N}{(N-1)!},$$

where C_n is independent of N. It is easy to show that for N large enough, this is less than 1. Thus $|\mathrm{LHS}| < 1$ for N large enough.

3. If $h(x) = g(x)(x-a)^N/N!$, where g is any polynomial with integer coefficients and $a \in \mathbb{Z}$, then the derivatives $h^{(j)}(a) = 0$ for $0 \leq j < N$ and in general $h^{(j)}(a) \in \mathbb{Z}$ for all $j \geq 0$. Since $f(x)/N$ has this form with $a \in \{1, \ldots, n\}$, it follows that $f^{(j)}(k)$ is an integer and is divisible by N, for all $j \geq 0$ and for $k \in \{1, \ldots, n\}$. Thus N divides all terms on the RHS of (2) having $k \neq 0$.

4. It remains to consider the terms with $k = 0$. Note that f has the form

$$f(x) = \sum_{j=N-1}^{m} \frac{c_j x^j}{(N-1)!}$$

where $c_{N-1} = (\pm n!)^N$ and $c_j \in \mathbb{Z}$ for all j. Then $f^{(j)}(0) = 0$ for $j < N-1$, $f^{(N-1)}(0) = c_{N-1}$, and $f^{(j)}(0) = c_j j!/(N-1)!$ for $j \geq N$ so N divides $f^{(j)}(0)$ if $j \neq N-1$.

5. The only term remaining on the RHS of (2) is $a_0 f^{(N-1)}(0) = a_0(\pm n!)^N$. This term is *not* divisible by N if N is prime with $N > |a_0|n$. Thus, we may choose N so that $|\mathrm{LHS}| < 1$ and so that in the RHS of (2), N divides every term $a_k f^{(j)}(k)$ except for $a_0 f^{(N-1)}(0)$. Therefore the RHS is a nonzero integer, so $|\mathrm{RHS}| \geq 1$.

This contradiction proves that e is not algebraic. Hence e is transcendental.