21-236 Analysis Assignment 4

Problems due Wednesday April 30:

4.1. (Pugh p358 #56) Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth. Prove that the pull-back T^* acts linearly on forms, and is natural with respect to composition in the sense that $(T \circ S)^* = S^* \circ T^*$. (Don't worry about being succinct—be clear.)

4.2. (Pugh p359 #59) Show that a smooth map $T\colon U \to V$ induces a linear map of cohomology groups $H^k(U) \to H^k(V)$ defined by (overloaded notation)

$$T^* : [\omega] \mapsto [T^*\omega]$$

Here, given ω in the vector space $Z^k(U)$ of closed k-forms on U, $[\omega]$ denotes the equivalence class of ω modulo the vector space $B^k(U)$ of exact k-forms on U. ($H^k(U)$ is the quotient space $Z^k(U)/B^k(U)$.)

4.3. (Pugh p360 #62) In $\mathbb{R}^3$ let $U = \{(x, y, z) : a < r < b\}$ be a spherical shell ($r = \sqrt{x^2 + y^2 + z^2}$ and $0 < a < b$). Show that the 2-form

$$\omega = \frac{x}{r^3} dy \wedge dz + \frac{y}{r^3} dz \wedge dx + \frac{z}{r^3} dx \wedge dy$$

is closed but not exact on U.

4.4. (a) Assume $G : M \to M$ is continuous, where M is homeomorphic to the ball

$$B_n = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| \leq 1\}.$$

Using Brouwer's fixed point theorem, prove that G has a fixed point.
(b) Given $\boldsymbol{x} \in \mathbb{R}^n$, we write $\boldsymbol{x} \geq 0$ if its components $x_j \geq 0$ for all j. Let

$$M_n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : |\boldsymbol{x}| = 1 \text{ and } \boldsymbol{x} \geq 0\}.$$

Prove that M_n is homeomorphic to B_n. (Suggestion: Using both Euclicean norm $|\boldsymbol{x}|$ and max norm $\|\boldsymbol{x}\|_\infty$, show B_n is homeomorphic to $[-1, 1]^n$, and M_n is homeomorphic to $[0, 1]^n$.)
(c) Suppose $A = (a_{ij})$ is an $m \times m$ matrix whose entries are all positive: $a_{ij} > 0$ for all i, j. Prove that A has a positive eigenvalue $\lambda > 0$ with corresponding eigenvector $\boldsymbol{v}$ having all positive components, by using parts (a) and (b) with

$$G(\boldsymbol{x}) = A\boldsymbol{x}/|A\boldsymbol{x}|.$$

4.5. (Pugh p360 #64) Consider forms on any open set $U \subset \mathbb{R}^n$. (a) Is the wedge product of closed forms necessarily closed? Of exact forms exact? What about the wedge product of a closed form and an exact form? (b) Try to describe a ring determined by the cohomology groups $H^1(U), \ldots, H^n(U)$, whose multiplication involves the wedge product. (Note $H^{n+1}(U) = 0$.)