

Problems due Monday Feb. 21:

3.1. Let E be a finite-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{R}$. Given f_1, \dots, f_n in E define an $n \times n$ real matrix $A = (a_{ij})$ by $a_{ij} = \langle f_i, f_j \rangle$.

(a) Show that A is positive definite if and only if f_1, \dots, f_n are linearly independent.

(b) If $H = (h_{ij})$ is defined by $h_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, \dots, n$, show that H is positive definite.

(c) Given a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ and a positive integer m , show that there is a unique real vector of coefficients $\mathbf{a} = (a_0, \dots, a_m)$ that *minimizes*

$$\int_0^1 |f(x) - p(x)|^2 dx, \quad p(x) = a_0 + a_1x + \dots + a_mx^m,$$

the mean square distance from f to a polynomial of degree m . In particular show that $H\mathbf{a} = \mathbf{b}$ where \mathbf{b} is a vector determined by f , and find \mathbf{b} .

3.2. Suppose invertible matrix $A(t)$ and vectors $\mathbf{x}(t)$ and $\mathbf{b}(t)$ are nonzero C^1 functions of t and satisfy $A(t)\mathbf{x}(t) = \mathbf{b}(t)$. Show that

$$\frac{\|\mathbf{x}'\|}{\|\mathbf{x}\|} \leq \kappa \left(\frac{\|\mathbf{b}'\|}{\|\mathbf{b}\|} + \frac{\|A'\|}{\|A\|} \right)$$

where $\kappa = \|A^{-1}\| \|A\|$ is the *condition number* of matrix A . (This estimate is used to bound relative errors in solutions of linear systems by relative errors in data. The Hilbert matrix H in problem 1 has a famously huge condition number.)

3.3. Let $J \in \mathcal{M}(2n \times 2n)$ be a given $2n \times 2n$ matrix which is *invertible* and *skew*, meaning $J^T = -J$. A canonical example with $n \times n$ blocks is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

A $2n \times 2n$ real matrix A is called *symplectic* if $A^T J A = J$. Let $\text{Sp}_{2n}(\mathbb{R})$ be the set of real symplectic matrices. (These are the matrices that preserve the *symplectic form* $(x, y) \mapsto \omega(x, y) := y^T J x$, and it is not hard to check that $\text{Sp}_{2n}(\mathbb{R})$ forms a group under matrix multiplication.) Show that for some neighborhood U of the identity matrix I , the set $\text{Sp}_{2n}(\mathbb{R}) \cap U$ is a smooth manifold of dimension $n(2n+1)$, and find its tangent space E_1 at I . In particular, find complementary subspaces E_1 and E_2 in $\mathcal{M}(2n \times 2n)$ and show that every symplectic matrix A near I has the form

$$A = I + X + G(X)$$

where $X \in E_1$ and $G : E_1 \rightarrow E_2$ is a C^∞ function satisfying $DG_0 = 0$. (Think of the case $n = 1$ first if you like.)

3.4. Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 , and $x_0 \in \mathbb{R}$. Show that for (t, x, y) in some open neighborhood of the point $(0, x_0, g(x_0))$ in \mathbb{R}^3 , the implicit equation

$$y = g(x - ty)$$

has a unique solution $y = u(t, x)$ near $g(x_0)$, such that $(t, x) \mapsto u(t, x)$ is a C^1 function from some neighborhood of $(0, x_0)$ to \mathbb{R} . Show that u satisfies the partial differential equation (Riemann's equation)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad \text{with } u(0, x) = g(x).$$

The function $u(t, x)$ is constant along certain lines in the (t, x) plane—what lines?