

21-236 Analysis Assignment 3

**Problems due Wednesday April 2:**

**3.1.** Let  $x, y : [0, 1] \rightarrow \mathbb{R}$  be  $C^1$ . Use Fubini's theorem and the Cauchy-Schwarz inequality for the inner product of  $(|x'(t)|, |y'(t)|)$  with  $(|x'(s)|, |y'(s)|)$  to prove "Cham's inequality" that appeared in the problem seminar:

$$\left(\int_0^1 |x'(t)| dt\right)^2 + \left(\int_0^1 |y'(t)| dt\right)^2 \leq \left(\int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt\right)^2$$

(Maybe you don't exactly need Fubini's theorem?)

**3.2.** Pugh p354 #46ab

**3.3.** Use the change of variables formula with polar coordinates  $(r, \theta) \mapsto (x, y) = (r \cos \theta, r \sin \theta)$  to compute

$$\int_S \frac{y}{(x^2 + y^2)^{5/2}},$$

where  $S$  is the region in the plane defined by

$$S := \{(x, y) \in \mathbb{R}^2 : x > 1, y > 0, x^2 + y^2 < 4\}.$$

**3.4.** Use Fubini's theorem in  $\mathbb{R}^3$ , with coordinates labeled  $(x, y, z)$ , to evaluate the volume of the bounded region enclosed by the plane  $z = x$  and the paraboloid  $z = x^2 + y^2$ . (Remark: Some ways are easier than others.)

**3.5.** (a) For vectors  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbb{R}^n$ , the *wedge product*  $\mathbf{a} \wedge \mathbf{b}$  is the vector whose components are the determinants of all the  $2 \times 2$  submatrices of the  $n \times 2$  matrix  $B = [\mathbf{a}, \mathbf{b}]$ , whose first column is  $\mathbf{a}$  and second column is  $\mathbf{b}$ . (For  $n = 3$  this is the same as the cross product.) The submatrices are determined by integer pairs  $(i, j)$  with  $1 \leq i < j \leq n$ , which are convenient to denote by  $i \wedge j$ . Thus the components of the wedge product take the form

$$(\mathbf{a} \wedge \mathbf{b})_{i \wedge j} = \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix}.$$

Show that  $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$  and the map  $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \wedge \mathbf{b}$  is bilinear.

(b) The parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$  is the set

$$P := \{\mathbf{v} = s\mathbf{a} + t\mathbf{b} : s, t \in [0, 1]\}.$$

Suppose both vectors  $\mathbf{a}$  and  $\mathbf{b}$  lie in the  $x_1x_2$  plane (meaning  $a_j = b_j = 0$  for  $j > 2$ ). Show that the area of the parallelogram  $P$  in this plane is  $|\mathbf{a} \wedge \mathbf{b}|$ , the Euclidean norm of the wedge product.

Hint: Find a  $2 \times 2$  matrix  $A$  such that  $AR = \hat{P}$ , where  $R = [0, 1] \times [0, 1]$  and  $\hat{P} \subset \mathbb{R}^2$  is the restriction of  $P$  into  $\mathbb{R}^2$ :  $(v_1, v_2) \in \hat{P}$  iff  $(v_1, v_2, 0, \dots, 0) \in P$ . Then use the change of variables formula to find the area

$$|\hat{P}| = \int_{\hat{P}} 1.$$