

21-236 Analysis Assignment 2

Problems due Friday February 22:

2.1. (cf. Pugh p 349 #20) Let U be a connected open set in \mathbb{R}^n . Suppose $f: U \rightarrow \mathbb{R}$ is C^1 and satisfies $Df_p = 0$ for all $p \in U$. Show that f is constant.

2.2. Suppose $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^1 . Prove f is not injective. (Consider two cases, one from the previous problem.)

2.3. An $n \times n$ matrix A is called *unimodular* if $\det A = 1$. The *special linear group* $SL(n, \mathbb{R})$ is the group consisting of all $n \times n$ real unimodular matrices. Show that for some neighborhood U of the identity matrix I , the set $SL(n, \mathbb{R}) \cap U$ is a smooth manifold of dimension $n^2 - 1$, and find its tangent space E at I . In particular, show that for any $n \times n$ matrix V not in E , every unimodular A near I has the form

$$A = I + X + G(X)$$

where $X \in E$ and $G: E \rightarrow \text{span}\{V\}$ is a C^∞ function satisfying $DG_0 = 0$. (Think of the case $n = 2$ first if you like.)

2.4. Given a C^2 function $v: \mathbb{R}^n \rightarrow \mathbb{R}$, the *Laplacian* of v is defined as the trace of the Hessian:

$$\Delta v(x) = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}(x).$$

A C^1 map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called *conformal* if at each point $p \in \mathbb{R}^n$, the derivative Df_p is a scalar multiple of an orthogonal transformation, meaning the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right)_p = \mu(p)V(p), \quad \text{where } \mu(p) \in \mathbb{R} \text{ and } V^T V = I.$$

Suppose $v: \mathbb{R}^n \rightarrow \mathbb{R}$ is C^2 and $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is conformal. Let $u = v \circ f$. Show that for all x ,

$$\Delta u(x) = \mu(x)^2(\Delta v)(f(x)) + \nabla v(f(x))\Delta f(x).$$

(b) If $n = 2$, show that necessarily $\Delta f = 0$ (assuming f is C^2).

Additional problems to think about, from Pugh pp 345-360: # 21, 30, 32, 34, 41**.