

**Problems due Monday Feb. 7:**

**2.1.** Suppose that all components of an  $n \times n$  matrix  $A(t) = (a_{ij}(t))$  and a vector  $\mathbf{b}(t) = (b_i(t))$  are smooth functions of  $t \in \mathbb{R}$ . Assume that  $\det A(t) \neq 0$ .

(a) Show that for the linear system  $A(t)\mathbf{x}(t) = \mathbf{b}(t)$ , that is,

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & & a_{2n}(t) \\ \vdots & & \ddots & \vdots \\ a_{n1}(t) & & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix},$$

the solution  $\mathbf{x}(t) = (x_j(t))$  is a smooth function of  $t$ . (Suggestion: use cofactors, or Gaussian elimination and induction on  $n$ .)

(b) Describe a formula for the derivatives  $x'_j(t)$ ,  $j = 1, \dots, n$  in terms of the components of  $A$  and  $\mathbf{b}$  and their derivatives.

**2.2.** (cf. Pugh p349 #20) Suppose  $U \subset \mathbb{R}^n$  is open and connected, and  $f : U \rightarrow \mathbb{R}^m$  is differentiable.

(a) If  $Df_p = 0$  for all  $p \in U$ , prove  $f$  is constant.

(b) If  $Df_p$  is constant, prove that  $f$  is affine (i.e., linear plus constant).

**2.3.** Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a  $C^1$  function. Prove  $f$  is not injective.

(Suggestion: Make use of the previous problem. Consider all cases.)

**2.4.** Pugh p352 #35. (Smoothness of inversion on  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  via the Neumann series.)

**2.5.** Given a  $C^2$  function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$ , the *Laplacian* of  $v$  is defined as the trace of the Hessian:

$$\Delta v(x) = \sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}(x).$$

A  $C^1$  map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called *conformal* if at each point  $p \in \mathbb{R}^n$ , the derivative  $Df_p$  is a scalar multiple of an *orthogonal transformation*, meaning the Jacobian matrix

$$\left( \frac{\partial f_i}{\partial x_j} \right)_p = \mu(p)V(p), \quad \text{where } \mu(p) \in \mathbb{R} \quad \text{and} \quad V^T V = I.$$

(a) Suppose  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are both  $C^2$ , and  $f$  is conformal. Let  $u = v \circ f$ . Show that for all  $x$ ,

$$\Delta u(x) = \mu(x)^2(\Delta v)(f(x)) + \nabla v(f(x))\Delta f(x).$$

(b) If  $n = 2$ , show that necessarily  $\Delta f = 0$  (assuming  $f$  is  $C^2$ ).

Additional problems to think about, but not to turn in:

From Pugh pp.349–354: #23, 25, 29, 32, 41\*\*.