

**Problems due Monday January 24:**

**1.1.** Let  $A = (a_{ij})$  be an  $m \times n$  matrix. Different vector norms on  $V = \mathbb{R}^n$  and  $W = \mathbb{R}^m$  yield different values for the operator norm

$$\|A\| = \sup \left\{ \frac{\|A\mathbf{v}\|_{\mathbb{R}^m}}{\|\mathbf{v}\|_{\mathbb{R}^n}} : \mathbf{v} \in \mathbb{R}^n \setminus \{0\} \right\}.$$

(i) If we use the sum norm on  $V$  and the max norm on  $W$ ,

$$\|\mathbf{v}\|_1 = \sum_{j=1}^n |v_j|, \quad \|\mathbf{w}\|_\infty = \max_{1 \leq i \leq m} |w_i|,$$

then find an explicit expression for  $\|A\|$ .

(ii) Show that if we use the sum vector norm for both  $V$  and  $W$ ,

$$\|\mathbf{v}\|_1 = \sum_{j=1}^n |v_j|, \quad \|\mathbf{w}\|_1 = \sum_{i=1}^m |w_i|,$$

$$\text{then } \|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|.$$

(One approach involves showing  $\|\mathbf{w}\|_1 = \sup\{\langle \mathbf{w}, \mathbf{u} \rangle : \|\mathbf{u}\|_\infty \leq 1\}$ .)

**1.2.** Pugh p346: #9 (Use the operator norm defined using the Euclidean vector norm. Hint: You have to get around the fact that the operator norm is generally hard to compute.)

**1.3.** Let  $V$  and  $W$  both be the (infinite-dimensional) vector space of polynomials on  $\mathbb{R}$ , but with the different norms

$$\|p\|_W = \int_0^1 |p(x)| dx, \quad \|p\|_V = \max_{[0,1]} |p(x)|.$$

a. Show that the identity map  $T(p) = p$  is not continuous from  $W$  to  $V$ .

b. Show that the linear map given by differentiation,  $S(p) = p'$ , is not continuous from  $V$  to  $V$ .

**1.4.** Let  $V$  be the vector space of real  $2 \times 2$  matrices, with norm given by the matrix norm (for the Euclidean norm, say). Define  $F : V \rightarrow V$  by  $F(A) = A^2$ .

(a) Show that  $F$  is differentiable with  $DF_A(B) = AB + BA$ . (Recall that typically  $AB \neq BA$ .)

(b) Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that as linear maps from  $V$  to  $V$ ,  $DF_I$  is an isomorphism, but  $DF_J$  is not.

Find a basis for the null space of  $DF_J$ ,  $\{B : DF_J(B) = 0\}$ .

**1.5.** Let  $0 < a < b$  and let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by  $f = (x, y, z)$ ,  $g = w$  where

$$w = w(x, y, z) = xy - z^2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix} = \begin{pmatrix} (b + a \sin s) \cos t \\ (b + a \sin s) \sin t \\ a \cos s \end{pmatrix}$$

Use the chain rule to determine the  $1 \times 2$  matrix  $[\partial w / \partial s, \partial w / \partial t]$  that represents  $D(g \circ f)$ . Geometrically, what is the range of  $f$ ? What are the level sets of  $g$ ? (Level sets are the preimages of constants.) Draw a picture.

Additional problems to think about, but not to turn in:

I recommend you study these problems from Pugh:

On pp. 345–349: #6, 13, 20, 21, 24

**1.A1.** Let  $V$  and  $W$  be vector spaces, and suppose  $f : V \rightarrow W$ . Show that the graph

$$\Gamma = \{(v, w) \in V \times W : w = f(v)\}$$

is a linear subspace of  $V \times W$  if and only if  $f$  is a linear transformation.

**1.A2.** Let  $V$  be the normed vector space  $C_b(\mathbb{R}, \mathbb{R})$  consisting of bounded continuous functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , with the supremum norm. Consider the self-composition map  $F : V \rightarrow V$  given by  $F(f) = f \circ f$ . That is,

$$F(f)(x) = f(f(x)) \quad \text{for all } x \in \mathbb{R}.$$

Prove that if  $f$  is  $C^1$  with derivative  $f'$ , and if  $g$  is uniformly continuous, then  $\partial_f F(g)$ , the directional derivative of  $F$  at  $f$  in direction  $g$ , exists, and find a formula for it. (I can't quite tell if  $F$  is necessarily differentiable at  $f$ .)