

Problems due Monday January 28:

1.1. Let $A = (a_{ij})$ be an $m \times n$ matrix. Different vector norms on $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ yield different values for the operator norm

$$\|A\| = \sup \left\{ \frac{\|A\mathbf{v}\|_{\mathbb{R}^m}}{\|\mathbf{v}\|_{\mathbb{R}^n}} : \mathbf{v} \in \mathbb{R}^n \setminus \{0\} \right\}.$$

(i) If we use the sum norm on V and the max norm on W ,

$$\|\mathbf{v}\|_1 = \sum_{j=1}^n |v_j|, \quad \|\mathbf{w}\|_\infty = \max_{1 \leq i \leq m} |w_i|,$$

then find an explicit expression for $\|A\|$.

(ii) Show that if we use the sum vector norm for both V and W ,

$$\|\mathbf{v}\|_1 = \sum_{j=1}^n |v_j|, \quad \|\mathbf{w}\|_1 = \sum_{i=1}^m |w_i|,$$

$$\text{then } \|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|.$$

(One approach involves showing $\|\mathbf{w}\|_1 = \sup\{\langle \mathbf{w}, \mathbf{u} \rangle : \|\mathbf{u}\|_\infty \leq 1\}$.)

1.2. Let V be a normed vector space, and suppose we have two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V . We say these norms are *comparable* (some say ‘equivalent’) if there exist positive constants c_-, c_+ such that for all $v \in V$,

$$c_- \|v\|_2 \leq \|v\|_1 \leq c_+ \|v\|_2.$$

- Prove that being comparable defines an equivalence relation on norms.
- Prove that if V is finite dimensional, any two norms are comparable. (Suggestion: Recall how to prove that $T : V \rightarrow V$ has bounded stretch.)

1.3. Let V and W both be the (infinite-dimensional) vector space of polynomials on \mathbb{R} , but with the different norms

$$\|p\|_W = \int_0^1 |p(x)| dx, \quad \|p\|_V = \max_{[0,1]} |p(x)|.$$

- Show that the identity map $T(p) = p$ is not continuous from W to V .
- Show that the linear map given by differentiation, $S(p) = p'$, is not continuous from V to V .

1.4. Let $0 < a < b$ and let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined by $f = (x, y, z)$, $g = w$ where

$$w = w(x, y, z) = xy - z^2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix} = \begin{pmatrix} (b + a \sin s) \cos t \\ (b + a \sin s) \sin t \\ a \cos s \end{pmatrix}$$

Use the chain rule to determine the 1×2 matrix $[\partial w / \partial s, \partial w / \partial t]$ that represents $D(g \circ f)$. Geometrically, what is the range of f ? What are the level sets of g ? (Level sets are the preimages of constants.) Draw a picture.

Additional problems to think about, but not to turn in:

I recommend you study these problems from Pugh:

On pp. 345–349: #6, 13, 20, 21, 24

1.A1. Let V and W be vector spaces, and suppose $f : V \rightarrow W$. Show that the graph

$$\Gamma = \{(v, w) \in V \times W : w = f(v)\}$$

is a linear subspace of $V \times W$ if and only if f is a linear transformation.

1.A2. Let V be the vector space of real 2×2 matrices, with norm given by the matrix norm (for the Euclidean norm, say). Define $F : V \rightarrow V$ by $F(A) = A^2$.

(a) Show that F is differentiable with $DF_A(B) = AB + BA$.

(b) Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that as linear maps from V to V , DF_I is an isomorphism, but DF_J is not. Find the rank of DF_J .