On total disconnectedness

Let M be a metric space. A standard definition is:

Definition 0. M is totally disconnected if every non-empty connected subset of M is a singleton. (I.e., for every $S \subset M$, S non-empty and connected implies $\exists p \in M$ with $S = \{p\}$.)

There are two interpretations of the ambiguous definition Pugh gives on page 96:

Definition 1. M is TD_1 if for all $p \in M$ and for all $r > 0$ there exists a subset U of $B(p, r)$ such that $p \in U$ and U is clopen in M.

Definition 2. M is TD_2 if for all $p \in M$ and for all $r > 0$ there exists a proper subset U of $B(p, r)$ such that $p \in U \neq$ and U is clopen in $B(p, r)$.

Let $N = B(p, r)$ for some $p \in M$, $r > 0$. The difference between the definitions is that a set U may be closed in N but not in M, which happens when U contains all its limit points that belong to N but not all those in M. If U is closed in M then U is closed in N. And since N is open, U is open in N if and only if U is open in M. Hence if M is TD_1 then M is TD_2. But I have not been able to prove or disprove the converse.

What is true is the following:

Theorem (i) M is $TD_1 \implies M$ is totally disconnected.
(ii) M is totally disconnected $\implies M$ is TD_2 .

Proof. (i) Assume M is TD_1. Let $S \subset M$ be non-empty and suppose there exist $p, q \in S$, $p \neq q$. I claim S is disconnected. Let $r = d(p, q)/2 > 0$, $N = B(p, r)$. Since M is TD_1, there exists $U \subset N$ such that $p \in U$ and U is clopen in M. THen U^c is clopen too, and $S = S_1 \cup S_2$ where $S_1 = S \cap U$ and $S_2 = S \cap U^c$ are both open in S (by inheritance). Since $p \in U \subset N$ and $q \in N^c \subset U^c$, we have $p \in S_1$, $q \in S_2$, so both S_1 and S_2 are non-empty. Hence S is disconnected.

(ii) Suppose M is not TD_2. Then there exists $p \in M$ and $r > 0$ such that $N = B(p, r)$ has no proper subsets U with $p \in U$ and U clopen in N. Thus N is connected, and $N \neq \{p\}$ (which *is* clopen). Hence N is not a singleton. Hence M is not totally disconnected.