21-235 Math Studies: Problem Seminar

14. Let $n+2$ points be given in $\mathbb{R}^n$ such that no one of them is a convex combination of the others.

    (a) Find 6 points in $\mathbb{R}^4$ such that the line segment joining any two does not intersect the convex hull of the other four. (Luke's conjecture seems to be false in $\mathbb{R}^4$.)

    (b) Must there always be some grouping into two disjoint subsets whose convex hulls intersect?

15. Suppose A is a closed, bounded, convex set in $\mathbb{R}^n$ that contains $\mathbf{0} = (0, \ldots, 0)$, and define $\alpha : \mathbb{R}^n \to [0, \infty)$ by

$$\alpha(y) = \sup\{\langle x, y \rangle : x \in A\}.$$

    (a) Let $B = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq \alpha(y) \text{ for all } y \in S^{n-1}\}$. Show that $A = B$. Interpret geometrically.

    (b) For $t \geq 0$, let A_t be the set of $x \in \mathbb{R}^n$ whose Euclidean distance to A is less than or equal to t. (The sets A_t 'grow at speed 1'.) Associated with A_t is

$$\alpha_t(y) = \sup\{\langle x, y \rangle : x \in A_t\}.$$

    How is α_t related to α?

    (c) Suppose A is a convex polygon in the plane $\mathbb{R}^2$. Show the area $|A_t|$ is a polynomial in t and describe the coefficients. Is the result true for a more general class of convex sets in the plane?