21-235 Math Studies: Problem Seminar

4. Let $k \in \mathbb{N}$, let X be a finite set, and let $\mathcal{F}$ be a collection of k-element subsets of X.

   Example. Let $k = 3$ and X be the set of positions in a tic-tac-toe board:

1	2	3
4	5	6
7	8	9

   Let $\mathcal{F}$ be the collection of rows, columns and diagonals on this board; to be precise,

   $$\mathcal{F} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\{2,5,8\},\{3,6,9\},\{1,5,9\},\{3,5,7\}\}.$$

   Two players, Maker and Breaker, play a game in which they alternately claim members of the ground set X. Maker wins if she manages to claim all the elements in some set f belonging to $\mathcal{F}$ (the collection of "winning sets"). Breaker wins if Maker does not.

   (a) If Maker moves first and both Maker and Breaker play intelligently, who wins the 'Maker-Breaker' version of tic-tac-toe?

   (b) Prove that in general, if $|\mathcal{F}| < 2^{k-1}$ then Breaker has a winning strategy for the Maker-Breaker game defined by $\mathcal{F}$.