21-235 Math Studies: Problem Seminar

1. Suppose a binary operation is defined on a set T. Assume further that the operation is not associative. Consider the product

$$a_1 \cdot a_2 \cdot \dots \cdot a_n$$

where $a_1, \dots, a_n \in T$. Of course, we can only make sense of this product by introducing brackets that indicate the order in which products should be taken. How many ways can this be done? (E.g. if $n = 3$ then there are 2 ways and if $n = 4$ then there are 5 ways.)

2. Suppose S is a set of natural numbers that is closed under addition, and suppose S contains two numbers x and y that are relatively prime (like 5 and 9).

(a) Show that S contains all sufficiently large integers. I.e., show that there is some integer z such that every integer greater than z belongs to S.

(b) Can you identify the smallest number z that can be guaranteed to have this property?

3. We are interested in the real number x that satisfies the following equation:

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2.$$

Of course, we recognize that the exponent in the term on the left is equal to the entire left hand side; in other words, we have

$$x^2 = 2.$$

We conclude that $x = \sqrt{2}$.

This, however, is not the end of the story. We are also interested in the real value y that satisfies

$$y^{y^{y^{\cdot^{\cdot^{\cdot}}}}} = 4.$$

Following the reasoning above we have $y^4 = 4$, and $y = \sqrt{2}$.

How can we make sense of

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} ?$$