The Logical Chain Reaction
(aka Mathematical Induction)

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Introduction

Lots of interesting mathematical facts involve statements or formulas that are supposed to be valid in general "for every postitive integer n." A famous example is the formula for the sum of the first n postitive integers:

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

Let $A(n)$ stand for this assertion. The claim that "$A(n)$ is true for every postitive integer n" is a claim that an infinite number of assertions are true:

$$\begin{aligned}
A(1) \text{ is the assertion:} &\qquad 1 = \frac{1(1+1)}{2}, \\
A(2) \text{ is the assertion:} &\qquad 1+2 = \frac{2(2+1)}{2}, \\
A(3) \text{ is the assertion:} &\qquad 1+2+3 = \frac{3(3+1)}{2}, \\
&\vdots \\
A(37) \text{ is the assertion:} &\qquad 1+2+\ldots+37 = \frac{37(37+1)}{2}, \\
A(38) \text{ is the assertion:} &\qquad 1+2+\ldots+38 = \frac{38(38+1)}{2}, \\
&\vdots
\end{aligned}$$

It is easy to check that $A(3)$ is true, say, but not nearly quite so easy for $A(100)$. (Gauss famously solved a related problem in a few minutes as a schoolboy.) Life is not long enough to check them all one by one.

To prove that $A(n)$ is true in general, we can use a strategy that relies on establishing a *logical link* between the consecutive statements. For example, I claim that $A(38)$ *follows logically from* $A(37)$. The point is this. *Suppose* $A(37)$ *were true* (just hypothetically for now). Then it would follow that

$$\begin{aligned} 1+2+\ldots+38 &= (1+2+\ldots+37)+38 = \frac{37\cdot 38}{2}+38 \\ &= \frac{37}{2}\cdot 38 + \frac{2}{2}\cdot 38 = \frac{39}{2}\cdot 38, \end{aligned}$$

hence

$$1+2+\ldots+38 = \frac{38(38+1)}{2}.$$

That is, $A(38)$ would be true as a consequence. Thus, $A(37)$ *implies* $A(38)$. We write $A(37) \rightarrow A(38)$. This implication is purely hypothetical at this point, since we did nothing to determine whether $A(37)$ is really true or not.

There was really nothing special about the number 37 in this argument, was there? Replacing 37 by an arbitrary postitive integer "n", we can prove that no matter what particular number "n" might represent, $A(n)$ implies $A(n+1)$: Suppose $A(n)$ were true for some particular n. Then it would follow that the sum of the first $n+1$ positive integers would be

$$\begin{aligned} 1+2+\ldots+(n+1) &= (1+2+\ldots+n)+(n+1) = \frac{n(n+1)}{2}+(n+1) \\ &= \frac{n}{2}(n+1)+\frac{2}{2}(n+1) = \left(\frac{n+2}{2}\right)(n+1), \end{aligned}$$

hence

$$1+2+\ldots+(n+1) = \frac{(n+1)((n+1)+1)}{2}.$$

That is, $A(n+1)$ would be true as a consequence. This proves that, as a hypothetical matter, $A(n) \rightarrow A(n+1)$. Since n was arbitrary, this holds for any postitive integer n.

The logical link from any of the statements $A(n)$ to the next one $A(n+1)$ is thereby established. We have constructed an infinite chain of implications:

$$A(1) \rightarrow A(2) \rightarrow A(3) \rightarrow \ldots \rightarrow A(37) \rightarrow A(38) \rightarrow \ldots$$

ad infinitum. The fuse is set, hypothetically speaking.

Now we light the end of the fuse: We prove $A(1)$ is true! This just says $1 = \frac{1(1+1)}{2}$. It is obvious; the proof is trivial. Bang! It instantly follows that $A(2)$ is true, that $A(3)$ is true, *ad infinitum*. This is the logical chain reaction — it proves that $A(n)$ *is in fact* true for *every* postitive integer n.

The strategy we just used is extremely flexible. It works to prove many statements of the form "$A(n)$ for every postitive integer n." To prove such a statement, there are two steps:

(i) Prove that, as a hypothetical matter, $A(n) \to A(n+1)$, with an argument that works for any arbitrary postitive integer n. That is, *supposing* $A(n)$ to be true for some particular n, *deduce* $A(n+1)$. (This sets the fuse.)

(ii) Prove $A(1)$. (Lights the end of the fuse.)

It instantly follows that $A(n)$ is true for every postitive integer n. (People usually do the steps in the other order. But this way emphasizes the hypothetical nature of step (i), and produces a nice Bang!)

This method of proof is traditionally called *mathematical induction*. I find this a very unfortunate term. People often confuse it with "inductive reasoning" in the scientific method, which is something completely different. Also, I think that the phrase "logical chain reaction" helps focus attention on what is important about the method — the logical link between consecutive statements.

A second example

The *Fibonacci numbers* 1, 1, 2, 3, 5, 8, 13, … are defined by the following recursive procedure: For $n = 0, 1, 2, \ldots$, let a_n denote the nth Fibonacci number. Then $a_0 = 1$, $a_1 = 1$, and

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2. \tag{1}$$

Problem. Prove that

$$a_n \leq \left(\frac{5}{3}\right)^n \qquad \text{for every positive integer } n. \tag{2}$$

Proof. We let $A(n)$ stand for the assertion:

$$a_n \leq \left(\frac{5}{3}\right)^n \quad \text{and} \quad a_{n-1} \leq \left(\frac{5}{3}\right)^{n-1}. \tag{3}$$

(i) Let n stand for a particular (but arbitrary) integer $n \geq 1$. Suppose that $A(n)$ is true for this n. Our goal is to deduce $A(n+1)$, that is, deduce that

$$a_{n+1} \leq \left(\frac{5}{3}\right)^{n+1} \quad \text{and} \quad a_n \leq \left(\frac{5}{3}\right)^n. \tag{4}$$

The second inequality is part of $A(n)$ already—it follows trivially. It remains to deduce the first: Using (1), then the hypothesis $A(n)$, and then the fact that $8/5 < 5/3$, we deduce that

$$\begin{aligned} a_{n+1} = a_n + a_{n-1} \quad &\leq \quad \left(\frac{5}{3}\right)^n + \left(\frac{5}{3}\right)^{n-1} \\ &= \quad \left(\frac{5}{3}\right)^n \left(1 + \frac{3}{5}\right) < \left(\frac{5}{3}\right)^n \left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^{n+1}, \end{aligned}$$

This shows that $A(n) \to A(n+1)$, for all positive integers n.

(ii) For $n = 1$, $a_1 = 1$ and $a_0 = 1$. We have $1 < 5/3 = (5/3)^1$ and $1 = (5/3)^0$, so indeed $a_1 \leq (5/3)^1$ and $a_0 \leq (5/3)^0$. That is, $A(1)$ is true.

By LCR (induction), we infer that $A(n)$ is in fact true for every integer $n \geq 1$. In particular, $a_n \leq (5/3)^n$ is true for every integer $n \geq 1$.

Problems

(LCR = Logical Chain Reaction)

1. Guess a general formula for the sum of the first n odd positive integers, and prove it using LCR (induction):

$$\begin{aligned} A(1):&\quad 1=1,\\ A(2):&\quad 1+3=4,\\ &\vdots\\ A(n):&\quad 1+3+5+\ldots+(2n-1)=\ ? \end{aligned}$$

2. Using LCR, prove that the following statements are true for every integer $n \geq 1$:

(a) $1^3+2^3+\ldots+n^3 < \dfrac{(n+1)^4}{4}$.

(b) $1^3+2^3+\ldots+n^3 = (1+2+\ldots+n)^2$.

3. Guess a general law which simplifies the product

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

and prove it using LCR.

4. Using LCR prove that for any x, if $x \neq 1$ then

$$1+x+x^2+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$$

for every integer $n \geq 1$.

5. Suppose that numbers s_1, s_2, s_3 etc. are computed successively by the rule

$$s_{n+1} = s_n + \frac{1}{(n+1)^2}\ ,$$

starting with the number $s_1 = 0$. Using LCR prove that

$$s_n \leq 1 - \frac{1}{n}$$

for every integer $n \geq 1$. Why doesn't induction work to prove directly that $s_n < 1$ for every $n \geq 1$?

6. In the second example above, we proved that the nth Fibonacci number a_n satisfies $a_n \leq b^n$ for every positive integer n, where $b = 5/3$. What is the smallest value of b for which a similar induction proof works?

7. Let $g(x) = 2x - 1$. Then

$$\begin{aligned} (g \circ g)(x) &= 2(2x-1) - 1 = 4x - 3, \\ (g \circ g \circ g)(x) &= 2(4x-3) - 1 = 8x - 7. \end{aligned}$$

Guess a general formula for the n-fold composition $(g \circ g \circ \ldots \circ g)(x)$, and prove it using LCR.

8. Given that $\lim_{x \to \infty} e^x = \infty$, use LCR and l'Hôpital's rule to prove that for each positive integer n,

$$\lim_{x \to +\infty} \frac{e^x}{x^n} = \infty \ .$$

9. Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, and $x \in \mathbb{R}$ is arbitrary.

(a) Integrate by parts using

$$u = f^{(n+1)}(t), \quad v = \frac{-(x-t)^{n+1}}{(n+1)}$$

to show that the definite integral

$$\int_0^x f^{(n+1)}(t) \frac{(x-t)^n}{n!}\, dt = f^{(n+1)}(0) \frac{x^{n+1}}{(n+1)!} + \int_0^x f^{(n+2)}(t) \frac{(x-t)^{n+1}}{(n+1)!}\, dt.$$

(b) Using the result of part (a), and LCR, show that the following statement is true for every integer $n \geq 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \int_0^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt.$$

(This is one version of *Taylor's theorem.*)