

21-235 Analysis Assignment 6

Problems due Friday Nov. 19

6.1. (Pugh p197 #52) Given that f and g are Riemann integrable on $[a, b]$, prove that $\max(f, g)$ and $\min(f, g)$ are, where

$$\max(f, g)(x) = \max(f(x), g(x)), \quad \min(f, g)(x) = \min(f(x), g(x)).$$

6.2. (Pugh p193 #35) Assume that $\psi : [a, b] \rightarrow \mathbb{R}$ is continuously differentiable. A **critical point** of ψ is a real number x where the derivative $\psi'(x) = 0$. A **critical value** is a real number y such that $y = \psi(x)$ for at least one critical point x . Prove that the set of critical values has measure zero. (This is known as the Morse-Sard theorem in dimension one.)

6.3. A function $f : [a, b] \rightarrow \mathbb{R}$ is called *càdlàg* iff at every point it is right continuous and has a left limit, i.e., for all $x \in [a, b]$ we have that

$$f(x) = \lim_{t \rightarrow x^+} f(t), \quad \text{and} \quad f_-(x) = \lim_{t \rightarrow x^-} f(t) \quad \text{exists.}$$

- (a) Prove: if f is càdlàg, then f has at most countably many discontinuities. (Hint: Study $D_\kappa = \{x \in [a, b] : \text{osc}_x f \geq \kappa\}$ where $\kappa > 0$.)
- (b) Show that if f is càdlàg, then f is Riemann integrable on $[a, b]$.

(càdlàg functions are fundamental in the theory of stochastic processes. The term derives from the French phrase *continue à droite, limite à gauche*.)

6.4. (cf. Pugh p198 #62) Suppose that $-1 < a_k \leq 0$ for all $k \in \mathbb{N}$. Prove that the infinite product

$$\prod_{k=1}^{\infty} (1 + a_k) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k)$$

converges to a nonzero limit if and only if $\sum_{k=1}^{\infty} a_k$ converges.

6.5. (Pugh p199 #68) (products of series)

In addition, study these problems from Pugh, pp. 189-197: 28, 30, 34, 53, 70 (try it!)