

21-235 Analysis Assignment 5

Problems due Friday Oct. 22

5.1. In this week's CNA seminar on shape matching, the following nice characterization of *Hausdorff distance* was featured. Let M be a compact metric space, and let $\mathcal{C}(M)$ be the set of nonempty compact subsets of M . Given any $A \subset M$ and $t \geq 0$, the *t-fattening* of A is the set

$$A_t = \{p \in M : \text{dist}(p, A) \leq t\}.$$

(a) If A is compact and $p \in M$, show there exists $a \in A$ such that

$$d(p, a) = \text{dist}(p, A) := \inf\{d(p, q) : q \in A\}.$$

(b) If A , B , and C are compact subsets in M and $t, s \geq 0$ are such that $C \subset B_s$ and $B \subset A_t$, show that $C \subset A_{s+t}$.

(c) Show that if we define

$$d_H(A, B) = \inf\{t \geq 0 : A \subset B_t \text{ and } B \subset A_t\},$$

then d_H is a metric on $\mathcal{C}(M)$. Suggestion: Show the inf is a min.

(d) Show that for any compact $A \subset M$ and any $\varepsilon > 0$ there is a finite set $S \subset M$ such that $d_H(A, S) < \varepsilon$.

5.2. (Hausdorff distance, continued)

(a) Suppose that we have a sequence of nested nonempty compact sets in $\mathcal{C}(M)$: $A_1 \supset A_2 \supset \dots$. Prove there is a set $A \in \mathcal{C}(M)$ such that $d_H(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$.

(b) Prove that $\mathcal{C}(M)$ is compact.

5.3. (Pugh p119 #44) Suppose that M is a compact metric space and that \mathcal{U} is an open cover of M which is "redundant" in the sense that each $p \in M$ is contained in at least two members of \mathcal{U} . Show that \mathcal{U} has a finite subcover with the same property.

5.4. (Pugh p120 #52) Suppose that M is compact and \mathcal{C} is a collection of closed subsets of M such that every intersection of finitely many members of \mathcal{C} is non-empty. Prove that there is some point p that is a member of every $C \in \mathcal{C}$. (Hint: Consider the collection of open sets C^c , $C \in \mathcal{C}$.)

5.5. (Pugh p137 Pre-lim problem #14) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous. Prove that there are constants A and B such that $|f(x)| \leq A + B|x|$ for all $x \in \mathbb{R}$.

Also, study these problems from Pugh, pp. 115-130: 39, 42, 46, 50, 54, 117.