21-235 Analysis Assignment 5

Problems due Monday November 19:

5.1. (Pugh p192 #30) Prove that the interval $[0, 1]$ is not a zero set.
(Watch out! — this is not completely easy.)

5.2. (Pugh p193 #35) Assume that $\psi : [a, b] \to \mathbb{R}$ is continuously differentiable.
A **critical point** of ψ is a real number x where the derivative $\psi'(x) = 0$.
A **critical value** is a real number y such that $y = \psi(x)$ for at least one critical point x. Prove that the set of critical values has measure zero.
(This is known as the Morse-Sard theorem in dimension one.)

5.3. (Pugh p 196 #50) Suppose that f and g are Riemann integrable on $[a, b]$, and $f(x) < g(x)$ for all $x \in [a, b]$. Prove the strict inequality

$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx.$$

5.4. A function $f\colon [a, b] \to \mathbb{R}$ is called *càdlàg* iff at every point it is right continuous and has a left limit, i.e., for all $x \in [a, b]$ we have that

$$f(x) = \lim_{t \to x^+} f(t), \quad \text{and} \quad f_-(x) = \lim_{t \to x^-} f(t) \quad \text{exists.}$$

(a) Prove: if f is càdlàg, then f has at most countably many discontinuities.
(Hint: Study $D_\kappa = \{x \in [a, b] : \operatorname{osc}_x f \geq \kappa\}$ where $\kappa > 0$.)

(b) Show that if f is càdlàg, then f is Riemann integrable on $[a, b]$.

(càdlàg functions are fundamental in the theory of stochastic processes. The term derives from the French phrase *continue à droit, limite à gauche*.)

In addition, study these problems from Pugh, pp. 189-197: 28, 34, 47, 52, 53a