21-235 Analysis Assignment 2

Problems due Friday September 21:

Turn in well-written solutions for the following problems:

- Pugh p46 #26
- (cf. Pugh p46 #30) Suppose $f : [a, b] \to \mathbb{R}$ is increasing. (This means $\forall x, y \in [a, b]$, $x \leq y \implies f(x) \leq f(y)$).

  (a) For all $x \in (a, b)$, show that quantities

  $$f_-(x) = \sup\{f(z) : a < z < x\} \leq f_+(x) = \inf\{f(z) : x < z < b\}$$

  exist, and f is continuous at x if and only if $f_-(x) = f_+(x)$.

  (b) Show that for any $c > 0$, the set $\{x \in (a, b) : f_+(x) - f_-(x) > c\}$ is finite.

  (c) Prove that the set of points in (a, b) where f is discontinuous is countable.

- Prove the following statements:

  (a) Every subset of a countable set is countable

  (b) Suppose $B \subset A$, $B \sim A$, and $B \neq A$. Prove that A is infinite.

  (c) Let $X = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{Z}\}$ be the set of sequences of integers. X is the cross product of denumerably many copies of $\mathbb{Z}$. Show that $X \sim \mathcal{P}(\mathbb{Z})$, hence $X \sim \mathbb{R}$.

In addition, study these problems from Pugh, pp. 41–50: 27(a), 32, 38

Here is a further problem (not to turn in) to deepen your understanding of limsup:

- Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers, and let

$$L = \limsup_{n\to\infty} a_n = \inf_{k\in\mathbb{N}} \left(\sup_{n\geq k} a_n \right).$$

Establish the following:

(a) If $M > L$ then $a_n > M$ for only finitely many $n \in \mathbb{N}$.

(b) If $M < L$ then $a_n > M$ for infinitely many $n \in \mathbb{N}$.

(c) Show that the converses of the statements in (a) and (b) are false. What is true instead?

(d) Prove

$$\begin{aligned} L &= \sup\{M \mid a_n > M \text{ for infinitely many } n \in \mathbb{N}\} \\ &= \inf\{M \mid a_n > M \text{ for only finitely many } n \in \mathbb{N}\}. \end{aligned}$$

(e) There exists a subsequence (a_{n_k}) such that $a_{n_k} \to L$ as $k \to \infty$.