Theorem (Stirling's formula): As $n \to \infty$, $\dfrac{n!}{n^n e^{-n}\sqrt{n}} \to \sqrt{2\pi}$.

Proof (after R. Michael, *Amer. Math. Month.* vol. 109 (2002) pp. 388-390):

Step 1. (Motivation for a rescaling)

$$n! = \Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx.$$

The function $f_n(x) = x^n e^{-x}$ satisfies $f_n'(x) = (nx^{n-1} - x^n)e^{-x}$ and this is positive for $x < n$, negative for $x > n$. So f_n has a maximum at $x = n$ with value $f_n(n) = n^n e^{-n}$. Thus

$$n! = n^n e^{-n} \int_0^\infty \frac{f_n(x)}{f_n(n)} \, dx.$$

The graph of $f_n(x)/f_n(n)$ has a rather shallow maximum at $x = n$, since

$$\frac{f_n''(x)}{f_n(n)} = \frac{(n(n-1)x^{n-2} - 2nx^{n-1} + x^n)e^{-n}}{n^n e^{-n}} = -\frac{1}{n} \quad \text{at } x = n.$$

We rescale to "sharpen up" this maximum: Let

$$y = \frac{x-n}{\sqrt{n}}, \qquad g_n(y) = \frac{f_n(n + y\sqrt{n})}{f_n(n)} = \left(1 + \frac{y}{\sqrt{n}}\right)^n e^{-\sqrt{n}y}.$$

Then $g_n(0) = 1$, $g_n'(0) = 0$ and $g_n''(0) = (\sqrt{n})^2(-1/n) = -1$. The fact that $g_n''(0)$ is independent of n motivates the choice of $\sqrt{n}$ in defining y. The substitution $x = n + y\sqrt{n}$, $dx = \sqrt{n}\, dy$ in the integral yields

$$n! = n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty g_n(y) \, dy.$$

This finishes step 1.

We will use the fact that

$$\sqrt{2\pi} = \int_{-\infty}^\infty e^{-y^2/2} \, dy.$$

Thus we have a *new goal*: Prove that as $n \to \infty$,

$$\int_{-\sqrt{n}}^\infty g_n(y) \, dy \to \int_{-\infty}^\infty e^{-y^2/2} \, dy.$$

The strategy to prove this goal involves proving that for all y, $g_n(y) \to e^{-y^2/2}$ as $n \to \infty$, but this by itself is not good enough. We will need two steps to make bounds on $g_n(y)$, and two more steps to bound the difference between the integrals involved.

Step 2. We will prove that for any constant $M > 0$, if $|y| \le M \le \frac{1}{2}\sqrt{n}$ then

$$|g_n(y) - e^{-y^2/2}| \le e^{M^3/\sqrt{n}} - 1 \ =: \hat{E}(M,n) \to 0 \quad \text{as } n \to \infty.$$

Step 3. We will prove that if $|y| \ge 1$ then $0 \le g_n(y) \le e^{-|y|/4}$ (for all $n \ge 1$). Note $e^{-y^2/2} \le e^{-|y|/4}$ also.

Step 4. We bound the difference of the integrals by breaking it into pieces. Define $g_n(y) = 0$ for $y < -\sqrt{n}$ for convenience. Then

$$\begin{aligned} E_n &= \int_{-\sqrt{n}}^{\infty} g_n(y)\,dy - \int_{-\infty}^{\infty} e^{-y^2/2}\,dy \\ &= \int_{-M}^{M} (g_n(y) - e^{-y^2/2})\,dy + \left(\int_{-\infty}^{-M} + \int_{M}^{\infty} \right) g_n(y)\,dy \\ &\quad - \left(\int_{-\infty}^{-M} + \int_{M}^{\infty} \right) e^{-y^2/2}\,dy. \end{aligned}$$

Using the triangle inequality, the bound on the 4 tails where $|y| \ge M$ from step 3, and the bound from step 2, we find

$$\begin{aligned} |E_n| &\le \int_{-M}^{M} |g_n(y) - e^{-y^2/2}|\,dy + 4\int_{M}^{\infty} e^{-|y|/4}\,dy \\ &\le 2M \cdot \hat{E}(M,n) + 4 \cdot 4e^{-M/4}. \end{aligned}$$

Step 5. We prove $\lim_{n\to\infty} E_n = 0$ by a classic kind of two-step ε-N argument. Let $\varepsilon > 0$. First choose M (depending on ε) so that $16e^{-M/4} < \varepsilon/2$. Then choose N (depending on M and ε, so really only on ε) so that $n > N$ implies $2M\hat{E}(M,n) < \varepsilon/2$. Then for all $n > N$ we have $|E_n| < \varepsilon$. This proves the required limit.

It remains to prove the bounds in steps 2 and 3.

Lemma $\log(1+x) = x - \frac{1}{2}x^2 + E_2(x)$ where $|E_2(x)| \le |x|^3$ for $|x| \le 1/2$.

Proof. Since $1/(1+x) = 1 - x + x^2/(1+x)$, integrating gives us

$$\log(1+x) = x + \frac{1}{2}x^2 + E_2(x), \qquad E_2(x) = \int_0^x \frac{t^2}{1+t}\,dt.$$

For $x > 0$, $E_2(x) \le \int_0^x t^2\,dt = x^3/3$, and for $-1/2 \le x < t < 0$,

$$0 < \frac{t^2}{1+t} \le 2t^2$$

and so $|E_2(x)| \le \int_x^0 2t^2\,dt = 2|x|^3/3 \le |x|^3$.

We now prove the bound claimed in Step 2. We first compute that

$$
\begin{aligned}
\log g_n(y) &= n \log\left(1 + \frac{y}{\sqrt{n}}\right) - \sqrt{n}y \\
&= n\left(\frac{y}{\sqrt{n}} - \frac{1}{2}\left(\frac{y}{\sqrt{n}}\right)^2 + E_2\left(\frac{y}{\sqrt{n}}\right)\right) - \sqrt{n}y \\
&= -\frac{1}{2}y^2 + nE_2\left(\frac{y}{\sqrt{n}}\right)
\end{aligned}
$$

Using Lemma 1 we find that whenever $|y| \leq M \leq \frac{1}{2}\sqrt{n}$,

$$
\left|\log g_n(y) + \frac{1}{2}y^2\right| = n\left|E_2\left(\frac{y}{\sqrt{n}}\right)\right| \leq n\left|\frac{y}{\sqrt{n}}\right|^3 \leq \frac{M^3}{\sqrt{n}}.
$$

Next, using that $|e^x - 1| \leq e^{|x|} - 1$ even when $x < 0$, we compute

$$
|g_n(y) - e^{-y^2/2}| = e^{-y^2/2}|e^{\log g_n(y) + y^2/2} - 1| \leq 1 \cdot (e^{M^3/\sqrt{n}} - 1)
$$

as claimed in Step 2.

It remains to prove the bound claimed in Step 3. First, for $x > 0$,

$$
-\log(1+x) + x = \int_0^x \frac{t}{1+t}\,dt \geq \int_0^x \frac{t}{1+x}\,dt = \frac{\frac{1}{2}x^2}{1+x}.
$$

Using this with $x = y/\sqrt{n}$, whenever $n \geq 1$ and $y \geq 1$ we find

$$
\begin{aligned}
-\log g_n(y) &= n\left(-\log\left(1 + \frac{y}{\sqrt{n}}\right) + \frac{y}{\sqrt{n}}\right) \\
&\geq n\frac{\frac{1}{2}(y/\sqrt{n})^2}{1 + (y/\sqrt{n})} = \frac{\frac{1}{2}y^2}{1 + (y/\sqrt{n})} \geq \frac{\frac{1}{2}y^2}{y + y} = \frac{y}{4}.
\end{aligned}
$$

It follows $g_n(y) \leq e^{-y/4}$ for $y \geq 1$.

Finally, for $-1 < x < 0$, substituting $t = -s$, $dt = -ds$ we find

$$
-\log(1+x) + x = \int_0^x \frac{t}{1+t}\,dt = \int_0^{|x|} \frac{s}{1-s}\,ds \geq \int_0^{|x|} s\,ds = \frac{1}{2}|x|^2.
$$

hence with $x = y/\sqrt{n}$, when $y \leq -1$, as above we have

$$
-\log g_n(y) = n\left(-\log\left(1 + \frac{y}{\sqrt{n}}\right) + \frac{y}{\sqrt{n}}\right) \geq \frac{n}{2}\left|\frac{y}{\sqrt{n}}\right|^2 = \frac{|y|^2}{2} \geq \frac{|y|}{2}.
$$

Hence $g_n(y) \leq e^{-|y|/2}$. This suffices to prove the bound in Step 3, and concludes the proof of Stirling's formula.