Theorem (Stirling’s formula): As $n \to \infty$, \[ \frac{n!}{n^n e^{-n} \sqrt{n}} \to \sqrt{2\pi}. \]


**Step 1.** (Motivation for a rescaling)

\[ n! = \Gamma(n + 1) = \int_0^\infty e^{-x} x^n \, dx. \]

The function \( f_n(x) = x^n e^{-x} \) satisfies \( f_n'(x) = (nx^{n-1} - x^n) e^{-x} \) and this is positive for \( x < n \), negative for \( x > n \). So \( f_n \) has a maximum at \( x = n \) with value \( f_n(n) = n^n e^{-n} \). Thus

\[ n! = n^n e^{-n} \int_0^\infty \frac{f_n(x)}{f_n(n)} \, dx. \]

The graph of \( f_n(x)/f_n(n) \) has a rather shallow maximum at \( x = n \), since

\[ \frac{f_n''(x)}{f_n(n)} = \frac{n(n-1)x^{n-2} - 2nx^{n-1} + x^n}{n^n e^{-n}} = -\frac{1}{n} \quad \text{at } x = n. \]

We rescale to “sharpen up” this maximum: Let

\[ y = \frac{x - n}{\sqrt{n}}, \quad g_n(y) = \frac{f_n(n + y\sqrt{n})}{f_n(n)} = \left(1 + \frac{y}{\sqrt{n}}\right)^n e^{-\sqrt{n}y}. \]

Then \( g_n(0) = 1 \), \( g_n'(0) = 0 \) and \( g_n''(0) = (\sqrt{n})^2 (-1/n) = -1 \). The fact that \( g_n''(0) \) is independent of \( n \) motivates the choice of \( \sqrt{n} \) in defining \( y \). The substitution \( x = n + y\sqrt{n}, \, dx = \sqrt{n} \, dy \) in the integral yields

\[ n! = n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty g_n(y) \, dy. \]

This finishes step 1.

We will use the fact that

\[ \sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-y^2/2} \, dy. \]

Thus we have a new goal: Prove that as \( n \to \infty \),

\[ \int_{-\sqrt{n}}^\infty g_n(y) \, dy \to \int_{-\infty}^{\infty} e^{-y^2/2} \, dy. \]

The strategy to prove this goal involves proving that for all \( y \), \( g_n(y) \to e^{-y^2/2} \) as \( n \to \infty \), but this by itself is not good enough. We will need two steps to make bounds on \( g_n(y) \), and two more steps to bound the difference between the integrals involved.
Step 2. We will prove that for any constant $M > 0$, if $|y| \leq M \leq \frac{1}{2} \sqrt{n}$ then

$$|g_n(y) - e^{-y^2/2}| \leq e^{M^2/\sqrt{n}} - 1 =: \hat{E}(M, n) \to 0 \quad \text{as } n \to \infty.$$ 

Step 3. We will prove that if $|y| \geq 1$ then $0 \leq g_n(y) \leq e^{-|y|/4}$ (for all $n \geq 1$). Note $e^{-y^2/2} \leq e^{-|y|/4}$ also.

Step 4. We bound the difference of the integrals by breaking it into pieces. Define $g_n(y) = 0$ for $y < -\sqrt{n}$ for convenience. Then

$$E_n = \int_{-\sqrt{n}}^{\infty} g_n(y) \, dy - \int_{-\infty}^{\infty} e^{-y^2/2} \, dy$$

$$= \int_{-M}^{M} (g_n(y) - e^{-y^2/2}) \, dy + \left( \int_{-\infty}^{-M} + \int_{M}^{\infty} \right) g_n(y) \, dy$$

$$- \left( \int_{-\infty}^{-M} + \int_{M}^{\infty} \right) e^{-y^2/2} \, dy.$$

Using the triangle inequality, the bound on the 4 tails where $|y| \geq M$ from step 3, and the bound from step 2, we find

$$|E_n| \leq \int_{-M}^{M} |g_n(y) - e^{-y^2/2}| \, dy + \int_{M}^{\infty} e^{-|y|/4} \, dy$$

$$\leq 2M \cdot \hat{E}(M, n) + 4 \cdot 4e^{-M/4}.$$ 

Step 5. We prove $\lim_{n \to \infty} E_n = 0$ by a classic kind of two-step $\varepsilon$-$N$ argument. Let $\varepsilon > 0$. First choose $M$ (depending on $\varepsilon$) so that $16e^{-M/4} < \varepsilon/2$. Then choose $N$ (depending on $M$ and $\varepsilon$, so really only on $\varepsilon$) so that $n > N$ implies $2M \hat{E}(M, n) < \varepsilon/2$. Then for all $n > N$ we have $|E_n| < \varepsilon$. This proves the required limit.

It remains to prove the bounds in steps 2 and 3.

Lemma $\log(1 + x) = x - \frac{1}{2} x^2 + E_2(x)$ where $|E_2(x)| \leq |x|^3$ for $|x| \leq 1/2$.

Proof. Since $1/(1 + x) = 1 - x + x^2/(1 + x)$, integrating gives us

$$\log(1 + x) = x + \frac{1}{2} x^2 + E_2(x), \quad E_2(x) = \int_{0}^{x} \frac{t^2}{1 + t} \, dt.$$ 

For $x > 0$, $E_2(x) \leq \int_{0}^{x} t^2 \, dt = x^3/3$, and for $-1/2 \leq x < t < 0$,

$$0 < \frac{t^2}{1 + t} \leq 2t^2$$

and so $|E_2(x)| \leq \int_{x}^{0} 2t^2 \, dt = 2|x|^3/3 \leq |x|^3$. 

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We now prove the bound claimed in Step 2. We first compute that
\[
\log g_n(y) = n \log \left(1 + \frac{y}{\sqrt{n}}\right) - \sqrt{ny} \\
= n \left(\frac{y}{\sqrt{n}} - \frac{1}{2} \left(\frac{y}{\sqrt{n}}\right)^2 + E_2\left(\frac{y}{\sqrt{n}}\right)\right) - \sqrt{ny} \\
= -\frac{1}{2} y^2 + nE_2\left(\frac{y}{\sqrt{n}}\right)
\]
Using Lemma 1 we find that whenever \(|y| \leq M \leq \frac{1}{2} \sqrt{n},\)
\[
\left|\log g_n(y) + \frac{1}{2} y^2\right| = n \left|E_2\left(\frac{y}{\sqrt{n}}\right)\right| \leq n \left|\frac{y}{\sqrt{n}}\right|^3 \leq \frac{M^3}{\sqrt{n}}.
\]
Next, using that \(|e^x - 1| \leq e|x| - 1\) even when \(x < 0,\) we compute
\[
|g_n(y) - e^{-y^2/2}| = e^{-y^2/2} |e^{\log g_n(y) + y^2/2 - 1}| \leq 1 \cdot (e^{M^2/\sqrt{n}} - 1)
\]
as claimed in Step 2.

It remains to prove the bound claimed in Step 3. First, for \(x > 0,\)
\[
-\log(1 + x) + x = \int_0^x \frac{t}{1+t} dt \geq \int_0^x \frac{t}{1+x} dt = \frac{1}{x} - \frac{1}{1+x}.
\]
Using this with \(x = y/\sqrt{n},\) whenever \(n \geq 1\) and \(y \geq 1\) we find
\[
-\log g_n(y) = n \left(-\log \left(1 + \frac{y}{\sqrt{n}}\right) + \frac{y}{\sqrt{n}}\right) \\
\geq n \left(\frac{1}{2} \left(\frac{y}{\sqrt{n}}\right)^2\right) = \frac{1}{2} y^2 (\frac{1}{1 + (y/\sqrt{n})}) \geq \frac{1}{2} y^2 \geq y/4.
\]
It follows \(g_n(y) \leq e^{-y^2/4}\) for \(y \geq 1.\)

Finally, for \(-1 < x < 0,\) substituting \(t = -s,\ dt = -ds\) we find
\[
-\log(1 + x) + x = \int_0^x \frac{t}{1+t} dt = \int_0^{|x|} \frac{s}{1-s} \frac{1}{s} ds = \int_0^{|x|} s ds = \frac{1}{2} |x|^2.
\]
hence with \(x = y/\sqrt{n},\) when \(y \leq -1,\) as above we have
\[
-\log g_n(y) = n \left(-\log \left(1 + \frac{y}{\sqrt{n}}\right) + \frac{y}{\sqrt{n}}\right) \geq n \frac{|y|}{2} \left(\frac{|y|}{2}\right) = \frac{|y|^2}{2} \geq \frac{|y|}{2}.
\]
Hence \(g_n(y) \leq e^{-|y|/2}.\) This suffices to prove the bound in Step 3, and concludes the proof of Stirling’s formula.