Tools for establishing convergence/divergence of series

1. Express the given series as a geometric or telescoping series or a finite value of the Riemann zeta function.

2. (Test for divergence) If the statement "$\lim_{n\to\infty} a_n = 0$" is false, then $\sum_{n=1}^{\infty} a_n$ diverges.

3. (Algebra of limits) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and $c \in \mathbb{R}$, then $\sum_{n=1}^{\infty}(a_n + b_n)$ and $\sum_{n=1}^{\infty} ca_n$ converge, and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \quad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

4. (Bounded partial sums) Suppose $a_n \geq 0$ for all $n \geq 1$. The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the sequence of partial sums $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is bounded.

5. (Integral test) Suppose $a_n \geq 0$ for all $n \geq 1$, and suppose $f : [1, \infty) \to \mathbb{R}$ is a continuous, decreasing function such that $f(n) = a_n$ for all $n \geq$ some N. Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(t)\, dt$ both converge or both diverge.

6. (Comparison test) Suppose $a_n \geq 0$ for all $n \geq 1$.

- If $0 \leq a_n \leq b_n$ for all $n \geq$ some N and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \geq$ some N and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

7. (Limit comparison test) Suppose $a_n \geq 0$ and $b_n > 0$ for all $n \geq 1$.

- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} > 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.
- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

8. (Alternating series test) If $a_n \downarrow 0$ as $n \to \infty$ (i.e., $0 \leq a_{n+1} \leq a_n$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = 0$), then the alternating series $\lim_{n\to\infty}(-1)^{n-1} a_n$ converges, and its sum S satisfies the inequalities

$$0 \leq (-1)^n \left(S - \sum_{k=1}^{n} (-1)^{k-1} a_k \right) \leq a_{n+1} \quad \text{for all } n \geq 1.$$

9. (Ratio test) Suppose $a_n \neq 0$ for all $n \geq 1$.

- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, then no conclusions can be drawn.

10. (Root test)

- If $\lim\limits_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim\limits_{n\to\infty} |a_n|^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim\limits_{n\to\infty} |a_n|^{1/n} = 1$, then no conclusions can be drawn.