21-132 Analysis II: Practice for Test 2 R. Pego

Test 2: 11:30 Friday, February 27, 2009

The material covered emphasizes chapter 7, chapter 9 and sections 10.1-10.3 from Apostol.

1. Suppose g is C^6 (six times continuously differentiable), c is fixed, and $f(x) = g(c+x) - 2g(c) + g(c-x)$. Find the Taylor polynomial $T_5 f(x)$ at $a = 0$, in terms of the derivatives of g at c.

2. Notice $\sqrt{2005} = 45\sqrt{1-x}$ where $x = 20/45^2$ satisfies $x = \frac{20}{2025} < \frac{1}{100}$ and $\frac{x}{1-x} = \frac{20}{2005} < \frac{1}{100}$. Using Taylor's formula with remainder, show that $|\sqrt{2005} - 44\frac{7}{9}| < 6 \times 10^{-4}$.

3. Find the following limits: $(a)\ \lim_{x\to\infty} \dfrac{x\log x}{e^x \log\log x}$, $\quad (b)\ \lim_{x\to 0} \dfrac{a^x - 1}{b^x - 1}\quad (b \neq 1)$.

4. Express the following complex numbers in the form $a + bi$.

$$(a)\quad \frac{2+3i}{3-4i}, \qquad (b)\quad \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{18}.$$

5. If a and b are integers, the complex number $z = a + ib$ is called a Gaussian integer. Find a Gaussian integer z such that $z\bar{z} = 5$. (This means 5 is not prime as a member of the set of Gaussian integers.)

6. Determine whether or not each sequence is convergent, and find the limit if it is:

$$(a)\left\{\left(1+\frac{3}{n}\right)^{4n}\right\} \qquad (b)\left\{2 + \frac{n}{n+1}\cos n\pi\right\}$$

7. Determine whether the increasing sequence $\{s_n\}$ defined by $s_n = \sum_{k=1}^{n} \dfrac{1}{k^2+1}$ is bounded, by comparison to an integral $\displaystyle\int_1^n \frac{1}{t^s}\,dt$ for some s.

8. Prove (using the ε-N definition of limit) that $\displaystyle\lim_{n\to\infty} \frac{-9n\sin n}{n^4 - 1} = 0$.

Topics:

Taylor polynomials, especially (see p. 287, and eq. (7.7)) for

$$e^x, \quad \sin x, \quad \cos x, \quad \sqrt{1+x}, \quad \frac{1}{1-x}, \quad \log(1-x)$$

Precise forms of the remainder term $E_n(x)$ in Taylor's formula.

L'Hopital's rule for 0/0 indeterminate forms $\lim_{x\to a} f(x)/g(x)$ and $\lim_{x\to\infty} f(x)/g(x)$.

Limits at $+\infty$ and infinite limits, including the precise definitions: E.g., $\lim_{x\to\infty} f(x) = -\infty$ means what exactly?

Complex arithmetic, polar form, complex exponentials.

Sequences: convergence & divergence, limits, establishing bounds by comparison.