Theorem: The series of reciprocal primes $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.

Proof (adapted from Wikipedia)

1. Fix n. By unique prime factorization, each integer j from 1 to n has a unique factorization $j = ab^2$ where $a, b \in \{1, \ldots, n\}$ and a is square-free, meaning it is the product of distinct primes. (a is the product of the primes that appear with odd powers in the prime factorization of j.)

2. Think of expanding the product

$$\prod_{p \leq n} \left(1 + \frac{1}{p}\right)$$

over all primes less than or equal n. We get all terms $1/a$ where a is square-free with all prime factors less than or equal n.

3. Multiply the product by the sum $\sum_{k=1}^{n} \frac{1}{k^2}$. By step 1 the terms $1/j$ for all j from 1 to n are included (plus more). Therefore

$$\sum_{j=1}^{n} \frac{1}{j} \leq \left(\sum_{k=1}^{n} \frac{1}{k^2}\right) \prod_{p \leq n} \left(1 + \frac{1}{p}\right)$$

4. Recall $\sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \int_{1}^{n} \frac{1}{t^2}\, dt \leq 2$ for all n, and $1 + x \leq e^x$ for all x.

Using $x = 1/p$, we get from step 3 that

$$\sum_{j=1} \frac{1}{j} \leq 2 \prod_{p \leq n} e^{1/p} = 2 \exp\left(\sum_{p \leq n} \frac{1}{p}\right).$$

It follows

$$\sum_{p \leq n} \frac{1}{p} \geq \log\left(\frac{1}{2} \sum_{j=1}^{n} \frac{1}{j}\right) \to \infty \quad \text{as } n \to \infty,$$

since the harmonic series diverges.