Nowhere differentiable continuous functions exist

(Adapted from an article by Liu Wen, "A nowhere differentiable continuous function constructed by infinite products" published in the American Mathematical Monthly, v. 109 (2002) pp. 378-380.)

We will construct nowhere differentiable continuous functions in the form of an infinite product of the form

$$f(x) = \prod_{n=1}^{\infty} (1 + a_n \sin \pi b_n x) \tag{1}$$

We suppose that $0 < a_n < 1$ for every n, and $\sum_{n=1}^{\infty} a_n$ converges. We suppose that b_n is a product $b_n = p_1 p_2 \cdots p_n$, where p_n is a positive even integer for each n, so $b_{n+1}/b_n = p_{n+1}$.

Theorem. Assume $\lim_{n\to\infty} n/(a_n p_n) = 0$. (Example: $a_n = \frac{1}{2} n^{-2}$, $p_n = 2n^4$.) Then the infinite product for $f(x)$ converges uniformly to a continuous function on $\mathbb{R}$, with the property that for all x, $f'(x)$ does not exist!!

1. The first step is to show that the log of the infinite product, namely the infinite series

$$\sum_{n=1}^{\infty} \log(1 + a_n \sin b_n \pi x) \tag{2}$$

converges uniformly, based on a comparison

$$|\log(1 + a_n \sin b_n \pi x)| \le C|a_n \sin b_n \pi x| \le C a_n$$

(C is from this week's homework.) By the Weierstrass M test the series (2) converges uniformly to a continuous function, hence the infinite product yields a continuous function $f(x)$ by exponentiation.

2. Let $x \in \mathbb{R}$ be arbitrary. We claim that

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$$

does not exist. For each $n \ge 1$, let $N_n = [b_n x]$, so $N_n/b_n \le x < (N_n + 1)/b_n$. Let

$$y_n = \frac{N_n + 1}{b_n}, \quad z_n = \frac{N_n + \frac{3}{2}}{b_n}.$$

Then $x < y_n < z_n$, $0 < b_n(z_n - x) \le 3/2$, and

$$\frac{1}{z_n - y_n} = 2b_n \le \frac{3}{z_n - x} \le \frac{3}{y_n - x}. \tag{3}$$

This implies

$$\begin{aligned} \frac{|f(z_n) - f(y_n)|}{z_n - y_n} &\le \frac{|f(z_n) - f(x)|}{z_n - y_n} + \frac{|f(y_n) - f(x)|}{z_n - y_n} \\ &\le 3\frac{|f(z_n) - f(x)|}{z_n - x} + 3\frac{|f(y_n) - f(x)|}{y_n - x}. \end{aligned}$$

If f is differentiable at x, then the right-hand side stays bounded as $n \to \infty$. We will show that the left-hand side is unbounded, which implies $f'(x)$ cannot exist.

3. Let $b = \prod_{n=1}^{\infty}(1+a_n)$, $\quad c = \prod_{n=1}^{\infty}(1-a_n)$, $\quad f_n(x) = \prod_{k=1}^{n}(1+a_k \sin b_k \pi x)$. Then the products b and c converge and $c < f_n(x) < b$ for all x. For $k \geq n$, $b_k y_n = (b_k/b_n)(N_n+1)$ is an integer, and for $k > n$, $b_k z_n$ is an integer, thus $\sin \pi b_k y_n = 0 = \sin \pi b_k z_n$ for these k. Also $\sin \pi b_n z_n = \sin(N_n + \frac{3}{2})\pi = (-1)^{N_n - 1}$. Hence

$$\begin{aligned} f(z_n) - f(y_n) &= f_n(z_n) - f_n(y_n) \\ &= f_{n-1}(z_n)(1 + a_n(-1)^{N_n-1}) - f_{n-1}(y_n)(1 + a_n \cdot 0). \\ &= (-1)^{N_n-1} a_n f_{n-1}(z_n) + (f_{n-1}(z_n) - f_{n-1}(y_n)) \end{aligned}$$

and because $f_{n-1}(z_n) \geq c$ we get

$$|f(z_n) - f(y_n)| \geq c a_n - |f_{n-1}(z_n) - f_{n-1}(y_n)|.$$

4. Define $h_k = 1 + a_k \sin \pi b_k z_n$, $\quad g_k = 1 + a_k \sin \pi b_k y_n$. For $k < n$ we have

$$|h_k - g_k| = a_k |\sin \pi b_k z_n - \sin \pi b_k y_n| \leq a_k b_k \pi (z_n - y_n) = \frac{a_k b_k \pi}{2 b_n} \leq \frac{\pi}{2 p_n}.$$

Now

$$f_{n-1}(z_n) - f_{n-1}(y_n) = \prod_{k=1}^{n-1} h_k - \prod_{k=1}^{n-1} g_k$$
$$= (h_1 - g_1) h_2 h_3 \cdots h_{n-1} + g_1 (h_2 - g_2) h_3 \cdots h_{n-1} + \ldots + g_1 g_2 \cdots g_{n-2}(h_{n-1} - g_{n-1})$$

(by converting the difference of products into a telescoping sum). Thus

$$|f_{n-1}(z_n) - f_{n-1}(y_n)| \leq \left(\prod_{k=1}^{n-1} (1 + a_k) \right) \sum_{k=1}^{n-1} |h_k - g_k| \leq b \frac{n\pi}{2 p_n}.$$

5. Combining steps 3 and 4 we get

$$|f(z_n) - f(y_n)| \quad \geq \quad c a_n - \frac{n\pi}{2p_n} = a_n \left(c - \frac{n}{a_n p_n} \frac{b\pi}{2} \right)$$

Since $n/(a_n p_n) \to 0$ as $n \to \infty$, from (3) follows

$$\frac{|f(z_n) - f(y_n)|}{z_n - y_n} \geq 2 b_n a_n \left(c - \frac{n}{a_n p_n} \frac{b\pi}{2} \right) \to \infty$$

as $n \to \infty$, since $b_n a_n \geq p_n a_n \to \infty$. By step 2, this shows $f'(x)$ does not exist. QED