21-132 Assignment 9: due Tuesday April 7

The **third test** will be held **Friday, April 10**, in class from 11:30-12:20. It will cover chapter 10: sequences, series, improper integrals.

(Problems in parentheses are recommended, but do not turn them in:)

9.1-2. From Apostol page 420, do problems 9, 21, (8, 10, 25)

9.3. From Apostol page 399, do problem 15

9.4. Suppose $\{a_n\}$ is a sequence decreasing to zero, and $f_n(x) = \dfrac{n}{1 + n^2(x - a_n)^2}$.

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$, first for all $x < 0$, then for all $x > 0$.

(b) Show that if $a < 0 < b$ then $\int_a^b f_n(x)\,dx \to \pi$ as $n \to \infty$.

(c) Finally, exhibit a sequence $\{a_n\}$ decreasing to zero such that *for all* x, $\lim_{n\to\infty} f_n(x) = 0$. (Thus $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b \lim_{n\to\infty} f_n(x)\,dx$.)

9.5. Let $f_n(x) = x^n$, so that for all $x \in (0,1)$, $f_n(x) \to 0$ as $n \to \infty$.

(a) Prove that $\{f_n\}$ does not converge to 0 uniformly on $(0,1)$.

(b) Let $b \in (0,1)$ be arbitrary. Prove that $\{f_n\}$ does converge to 0 uniformly on $(0,b)$.

9.6. Let $f : \mathbb{R} \to \mathbb{C}$ be given by $f(x) = e^{ix}$, so $f'(x) = ie^{ix}$. Show that the conclusion of the Mean Value Theorem for real-valued functions is false here. I.e., show that if $a < b$ then there is *no* $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$. (Hint: consider distance.)

9.7. The series $S = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}$ converges by the alternating series test. Prove S is irrational. (Suppose $S = \frac{m}{n}$. Try for a contradiction by (10.48).) (What is S? Can you prove your answer?)