

21-132 Assignment 2: due Tuesday January 27

Reading: Apostol, chapter 6.

(Problems in parentheses are recommended, but do not turn them in:)

2.1-4. From Apostol page 236, do problems 3, 8, 22, 31 (2, 7, 15, 35)

2.5. From Apostol page 242, do problem 3.

2.6-8. From Apostol page 248, do problems 24, 39, 41 (11, 27, 35, 42)

2.9. (Differentiability of inverse functions) Suppose $F : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing continuous function, and G is its inverse function: $G(y) = x$ if and only if $y = F(x)$. Suppose F is differentiable at a with $F'(a) > 0$. Prove G is differentiable at $b = F(a)$ with $G'(b) = 1/F'(a)$.

Use an ε - δ argument, filling in the GAPS in the following argument.

Proof. We must show

$$\lim_{y \rightarrow b} \frac{G(y) - G(b)}{y - b} = \frac{1}{F'(a)}. \quad (1)$$

Since F is differentiable at a and we know $F'(a) \neq 0$, we know from the basic limit theorems that

$$\lim_{x \rightarrow a} \frac{x - a}{F(x) - F(a)} = \frac{1}{F'(a)}. \quad (2)$$

Let $\varepsilon > 0$ be arbitrary. Then (2) means that there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \text{ implies } \left| \frac{x - a}{F(x) - F(a)} - \frac{1}{F'(a)} \right| < \varepsilon. \quad (3)$$

Claim: There exists $\delta_2 > 0$ such that for all y , $0 < |y - b| < \delta_2$ implies $0 < |x - a| < \delta_1$, where $x = G(y)$ and $a = G(b)$.

Proof of the claim: [GAP! You fill in the proof that δ_2 exists.]

Using the claim, we finish the proof as follows: Suppose $0 < |y - b| < \delta_2$. Let $x = G(y)$, then $0 < |x - a| < \delta_1$ and by (3) we have that

$$\left| \frac{G(y) - G(b)}{y - b} - \frac{1}{F'(a)} \right| = \left| [\text{GAP!}] - \frac{1}{F'(a)} \right| < \varepsilon.$$

This proves (1). QED