

21-132 Assignment 10: due Thursday April 23

The **fourth test** will be held **Wednesday, April 29**, in class from 11:30-12:20. It will focus on chapter 11 sections 11.1-11.11 and 11.14: sequences of functions, uniform convergence, power series, radius of convergence, integration and differentiation of power series, Taylor series, exponential and trig functions, power series and differential equations.

(Problems in parentheses are recommended, but do not turn them in:)

**10.1-2.** From Apostol page 430, do problems 3, 9, (15)

**10.3-6.** From Apostol page 438, do problems 7, 9, 13, 17

**10.7-9.** From Apostol page 443, do problems 2, 5, 11, (10, 15)

(In problem 11, show  $y = J_0(x)$  satisfies  $x^2y'' + xy' + x^2y = 0$ , and  $y = J_1(x)$  satisfies  $x^2y'' + xy' + (x^2 - 1)y = 0$ .)

**10.10.** (Convergence of infinite products) Suppose that  $|a_n| < 1$  for all  $n \geq 1$  and  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\alpha = \sup\{|a_n|\} < 1$ .

(a) Find a constant  $C$  with the property that

$$-C|a_n| \leq \log(1 + a_n \sin nx) \leq |a_n|$$

for all  $x \in \mathbb{R}$  and all  $n \geq 1$ .

(b) Deduce (from the appropriate test) that if  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then

$$f(x) = \sum_{n=1}^{\infty} \log(1 + a_n \sin nx)$$

converges uniformly on  $\mathbb{R}$ , and therefore the infinite product

$$p(x) = \prod_{n=1}^{\infty} (1 + a_n \sin nx) = e^{f(x)}$$

converges and is a continuous function of  $x$ .

• Here a couple of examples of famous infinite product formulas (They are not so easy to prove!):

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right),$$
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1} \quad \text{for all } s > 1,$$

where  $p_n$  is the  $n$ th prime number.