Fourier series — a first convergence theorem

Fourier series have the form

$$S(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

A Fourier series with finitely many nonzero terms is called a *trigonometric polynomial*, written

$$T_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx).$$

(We say such a T_N has degree N.) Since $\cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$ and $\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$, we can write $T_N(x)$ in a more elegant complex form:

$$T_N(x) = a_0 + \sum_{n=1}^{N}\left(\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-inx}\right) = \sum_{n=-N}^{N} c_n e^{inx}$$

where $c_0 = a_0$, and for $n > 0$, $c_n = \frac{1}{2}a_n + \frac{1}{2i}b_n$ and $c_{-n} = \frac{1}{2}a_n - \frac{1}{2i}b_n$.

Theorem *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic, and piecewise C^1, and define c_n by*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbb{Z}. \tag{2}$$

Then the series $S(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly on $\mathbb{R}$, and $S(x) = f(x)$ for all $x \in \mathbb{R}$.

The proof has three main steps (each of independent interest):

I. (Uniform approximation by trig polynomials) We show that for any $f : \mathbb{R} \to \mathbb{R}$ continuous and 2π-periodic, for any $\varepsilon > 0$ there exists a trig polynomial $\tilde{T}_N$ such that $|f(x) - \tilde{T}_N(x)| < \varepsilon$ for all x.

II. (Optimal approximation in square-integral sense) Given any $f : \mathbb{R} \to \mathbb{R}$ continuous and 2π-periodic, and given N, we exhibit a particular T_N with the property that

$$\int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - \tilde{T}_N(x)|^2\,dx \tag{3}$$

for all trig polynomials $\tilde{T}_N$ of degree $\le N$. Namely, $T_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$ where c_n is given by (2).

III. (Uniform convergence of the series) Given $f : \mathbb{R} \to \mathbb{R}$ continuous, 2π-periodic and piecewise C^1, we show that $\sum_{n=-\infty}^{\infty} |c_n|$ converges, and hence

$$S(x) = \lim_{N\to\infty} T_N(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges uniformly on $\mathbb{R}$ (by the Weierstrass M test).

Assuming these things are proved, let us deduce the Theorem. Given f as stated in the Theorem, we then deduce from II that the integral

$$\int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\,dx$$

decreases as N increases. In the limit $N \to \infty$ this approaches $\int_{-\pi}^{\pi} |f(x) - S(x)|^2\,dx$, by the uniform convergence of III. Then, for any trig polynomial $\tilde{T}_N$ of any degree, we have

$$\int_{-\pi}^{\pi} |f(x) - S(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x) - \tilde{T}_N(x)|^2\,dx. \tag{4}$$

By step I we infer that for any $\varepsilon > 0$ there exists $\tilde{T}_N$ such that the right-hand side is less than $2\pi\varepsilon^2$. Since $|f(x) - S(x)|^2$ is continuous and non-negative, it follows $S(x) = f(x)$ for all x. This proves the Theorem.

Proof of I, part a. Define functions u_n $(n = 1, 2, \ldots)$ by

$$u_n(x) = \frac{1}{\alpha_n}\left(\frac{1+\cos nx}{2}\right)^n, \qquad \alpha_n = \int_{-\pi}^{\pi}\left(\frac{1+\cos nx}{2}\right)^n dx. \tag{5}$$

We claim that these functions have the properties:

(i) For all $n \geq 1$ and all x, $u_n(x) \geq 0$ and $u_n(x + 2\pi) = u_n(x)$.

(ii) For all $n \geq 1$, $\int_{-\pi}^{\pi} u_n(x)\, dx = 1$.

(iii) For all $\delta \in (0, \pi)$, $\displaystyle\int_{\delta \leq |x| \leq \pi} u_n(x)\, dx \to 0$ as $n \to \infty$.

(i) and (ii) are clear, let us prove (iii). Fix $\delta \in (0, \pi)$, and note $x \mapsto 1 + \cos x$ is even, and decreasing on $[0, \pi]$. For any $x \in [\delta, \pi]$ and $y \in [0, \frac{1}{2}\delta]$,

$$\frac{1+\cos x}{1+\cos\delta} \leq 1 \leq \frac{1+\cos y}{1+\cos\frac{1}{2}\delta},$$

hence $1 + \cos x \leq r(1 + \cos y)$ where $r < 1$. Taking powers and dividing by α_n we get $u_n(x) \leq r^n u_n(y)$. Integrating over $y \in [0, \frac{1}{2}\delta]$ we infer

$$\int_0^{\delta/2} u_n(x)\, dy = \frac{\delta}{2} u_n(x) \leq r^n \int_0^{\delta/2} u_n(y)\, dy \leq r^n \cdot 1 \to 0$$

as $n \to \infty$. Thus $u_n(x) \to 0$ as $n \to \infty$, *uniformly for* $x \in [\delta, \pi]$, and (iii) follows.

Note: By expanding the power, we see u_N is a trig polynomial, with $u_N(x) = \sum_{n=-N}^{N} \beta_n e^{inx}$ for some constants β_n.

Proof of I, part b. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 2π-periodic. Define

$$(u_n * f)(x) = \int_{-\pi}^{\pi} u_n(y) f(x - y)\, dy. \tag{6}$$

1. We claim $u_n * f$ is a trig polynomial. Proof: Substitute $z = x - y$ and use 2π-periodicity to get

$$(u_N * f)(x) = \int_{x+\pi}^{x-\pi} u_N(x - z) f(z)\, (-dz) = \int_{-\pi}^{\pi} f(z) \sum_{n=-N}^{N} \beta_n e^{in(x-z)}\, dz = \sum_{n=-N}^{N} \beta_n e^{inx} \int_{-\pi}^{\pi} f(z) e^{-inz}\, dz$$

which shows $u_N * f$ is a trig polynomial $\sum_{n=-N}^{N} \tilde{c}_n e^{inx}$.

2. We claim $(u_N * f)(x) \to f(x)$ as $N \to \infty$, uniformly for $x \in [-\pi, \pi]$ (actually all x by 2π-periodicity). Proof: First, use property (ii) to write

$$(u_N * f)(x) - f(x) = \int_{-\pi}^{\pi} u_N(y)(f(x - y) - f(x))\, dy. \tag{7}$$

f is *uniformly* continuous on $[-2\pi, 2\pi]$ (a closed bounded interval, remember from last semester). Given $\varepsilon > 0$, therefore there exists $\delta > 0$ such that for all $x \in [-\pi, \pi]$ and y with $|y| < \delta$, $|f(x - y) - f(x)| < \varepsilon/2$. Then by breaking the integral in two parts, where $|y| < \delta$ and $\delta \leq |y| \leq \pi$, from (7) we get

$$|(u_N * f)(x) - f(x)| \leq \int_{|y|<\delta} u_N(y) \cdot \frac{\varepsilon}{2}\, dy + \int_{\delta \leq |y| \leq \pi} u_N(y) \cdot 2M\, dy,$$

where $M = \sup\{|f(y)|, y \in \mathbb{R}\} < \infty$. The first term on the right-hand side is less than $1 \cdot \varepsilon/2$ by property (ii), and the second term tends to 0 as $N \to \infty$ by property (iii). Hence there exists N_0 such that for all $N > N_0$, $|(u_N * f)(x) - f(x)| < \varepsilon$. This finishes the proof of I.

Proof of II. (Mean-square optimality) We introduce the notation

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx, \qquad e_n(x) = e^{inx}, \tag{8}$$

which has the properties

$$\begin{aligned}
(f, f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx, && (9)\\
(a_1 f_1 + a_2 f_2, g) &= a_1(f_1, g) + a_2(f_2, g), && (10)\\
(f, a_1 g_1 + a_2 g_2) &= \overline{a_1}(f, g_1) + \overline{a_2}(f, g_2), && (11)\\
(e_n, e_m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} && (12)
\end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ continuous and 2π-periodic be given. Fix N. Let

$$T_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} c_n e_n(x) \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = (f, e_n).$$

Then for any integer m with $|m| \leq N$,

$$(f - T_N, e_m) = (f, e_m) - \sum_{n=-N}^{N} c_n (e_n, e_m) = c_m - c_m = 0,$$

hence by (11), whenever $\tilde{T}_N = \sum_{n=-N}^{N} \tilde{c}_n e_n$ is a trig polynomial of degree $\leq N$,

$$(f - T_N, \tilde{T}_N) = 0. \tag{13}$$

In particular, taking $\tilde{T}_N = T_N$ we compute that

$$\begin{aligned}
(f, f) &= ((f - T_N) + T_N, (f - T_N) + T_N) \\
&= (f - T_N, f - T_N) + (T_N, f - T_N) + (f - T_N, T_N) + (T_N, T_N) \\
&= (f - T_N, f - T_N) + (T_N, T_N)
\end{aligned}$$

because the cross terms are zero. This proves a *generalized Pythagorean theorem*:

$$\int_{-\pi}^{\pi} |f(x)|^2\, dx = \int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\, dx + \int_{-\pi}^{\pi} |T_N(x)|^2\, dx. \tag{14}$$

Now, for any *other* trig polynomial $\tilde{T}_N = \sum_{n=-N}^{N} \tilde{c}_n e_n$ of degree $\leq N$, doing a similar expansion and using the fact that $T_N - \tilde{T}_N$ is a trig polynomial, we deduce that

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \tilde{T}_N(x)|^2\, dx &= (f - T_N + T_N - \tilde{T}_N, f - T_N + T_N - \tilde{T}_N) \\
&= (f - T_N, f - T_N) + (T_N - \tilde{T}_N, T_N - \tilde{T}_N) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\, dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} |T_N(x) - \tilde{T}_N(x)|^2\, dx \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - T_N(x)|^2\, dx.
\end{aligned}$$

This proves the optimality property II.

Also note that

$$(T_N, T_N) = \Big(\sum_{n=-N}^{N} c_n e_n, \sum_{m=-N}^{N} c_m e_m\Big) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} (e_n, e_m) = \sum_{n=-N}^{N} |c_n|^2.$$

Using this in (14) proves *Bessel's inequality*

$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx. \tag{15}$$

Hence $\sum_{n=-\infty}^{\infty} |c_n|^2$ converges. (Note that for this argument, f need not be continuous, only integrable.)

Proof of III. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, 2π-periodic and piecewise C^1, and define c_n by (2) for $n \in \mathbb{Z}$. We give the proof for the case that for some $a \in [-\pi, \pi]$, f is C^1 on $[-\pi, a]$ and $[a, \pi]$, meaning f is continuous on these intervals and continuously differentiable on $(-\pi, a)$ and (a, π), so the fundamental theorem of calculus applies. (With more discontinuities of f' the notation is just more cumbersome.) Then

$$2\pi c_n = \int_{-\pi}^{a} f(x) e^{-inx} \, dx + \int_{a}^{\pi} f(x) e^{-inx} \, dx.$$

Integrating by parts we get

$$\begin{aligned} \int_{-\pi}^{a} f(x) e^{-inx} \, dx &= f(x) \frac{e^{-inx}}{-in} \Big|_{-\pi}^{a} - \int_{-\pi}^{a} f'(x) \frac{e^{-inx}}{-in} \, dx \\ &= \frac{f(a) e^{-ina} - f(-\pi) e^{in\pi}}{-in} + \int_{-\pi}^{a} f'(x) \frac{e^{-inx}}{in} \, dx. \end{aligned}$$

Similarly,

$$\int_{a}^{\pi} f(x) e^{-inx} \, dx = \frac{f(\pi) e^{-in\pi} - f(a) e^{-ina}}{-in} + \int_{a}^{\pi} f'(x) \frac{e^{-inx}}{in} \, dx.$$

Adding, and using $f(\pi) = f(-\pi)$ and $e^{in\pi} = e^{-in\pi}$ we find that

$$c_n = \frac{1}{2\pi in} \int_{-\pi}^{\pi} f'(x) e^{-inx} \, dx = \frac{1}{in} (f', e_n) = \frac{\gamma_n}{in}, \quad \text{where } \gamma_n = (f', e_n).$$

By Bessel's inequality for f' we have

$$\sum_{n=-N}^{N} |\gamma_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx.$$

By the Cauchy-Schwartz inequality (Apostol I.41),

$$\begin{aligned} \sum_{1 \leq |n| \leq N} |c_n| = \sum_{1 \leq |n| \leq N} \frac{1}{n} |\gamma_n| &\leq \left(\sum_{1 \leq |n| \leq N} \left(\frac{1}{n} \right)^2 \right)^{1/2} \left(\sum_{1 \leq |n| \leq N} |\gamma_n|^2 \right)^{1/2} \\ &\leq (2\zeta(2))^{1/2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx \right)^{1/2} \end{aligned}$$

which is constant independent of N. Therefore

$$\lim_{N \to \infty} \sum_{n=-N}^{N} |c_n| = \sum_{n=-\infty}^{\infty} |c_n|$$

converges. This proves III, and the Theorem follows.