Theorem: $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Proof (after R. Michael, *Amer. Math. Month.* vol. 109 (2002) pp. 388-390): Compute that

$$\frac{d}{dt}\left(\int_0^t e^{-x^2}\,dx\right)^2 = 2\left(\int_0^t e^{-x^2}\,dx\right)e^{-t^2} = \int_0^1 e^{-t^2y^2}2te^{-t^2}\,dy$$
$$= -\int_0^1 \frac{d}{dt}\left(\frac{e^{-t^2(y^2+1)}}{y^2+1}\right)dy = -\frac{d}{dt}\int_0^1\left(\frac{e^{-t^2(y^2+1)}}{y^2+1}\right)dy$$

In the second step we used the substitution $x = ty$, $dx = t\,dy$. The last step (interchanging order of differentiation and integration) requires a careful justification that we do not do here. But now, if we define

$$f(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2 + \int_0^1\left(\frac{e^{-t^2(y^2+1)}}{y^2+1}\right)dy,$$

we find that $f'(t) = 0$ for all t, $-\infty < t < \infty$. Hence for all t,

$$f(t) = f(0) = \int_0^1 \frac{1}{y^2+1}\,dy = \arctan 1 = \frac{\pi}{4}.$$

In the limit $t \to \infty$, on the other hand, we have a simple comparison:

$$0 \le \int_0^1 \frac{e^{-t^2(y^2+1)}}{y^2+1}\,dy \le \int_0^1 \frac{e^{-t^2(0+1)}}{0+1}\,dy = e^{-t^2} \to 0.$$

Therefore

$$\frac{\pi}{4} = \lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(\int_0^t e^{-x^2}\,dx\right)^2 = \left(\lim_{t\to\infty}\int_0^t e^{-x^2}\,dx\right)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)^2,$$

and the claimed result follows.

Remark: An important function in probability theory and many applications is the *error function* defined by

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx.$$

By the result above, $\lim_{t\to\infty}\mathrm{erf}(t) = 1$.