

Force of gravity between the earth and moon

The story is that Newton took a long time (almost 20 years) to publish his theory of gravity, that the force between two point masses m and M goes as the inverse square of the distance D between them:

$$F = \frac{GMm}{D^2}.$$

Scholars have argued that the reason for the delay was that Newton was not happy with his argument that the force between solid spheres goes as the inverse square of the distance between their centers. After all, the earth and moon are evidently not point masses—just look down!

The calculation of the force between a solid ball of radius A and a point mass m at distance $R > A$ from the center of the ball can be done with integrals as follows. Introduce (x, y, z) coordinates with origin at the center of the ball and point mass m at $(R, 0, 0)$ on the x axis. By symmetry, the net force will be aligned along the x axis.

Cut up the ball into small pieces as follows: Cut the ball into spherical shells of thickness Δr . Divide each shell into rings subtending angle $\Delta\theta$ with θ measuring the polar angle from the positive x axis. Finally, cut each ring in chunks subtending angle $\Delta\phi$ measured around the x axis.

We use polar coordinates (r, θ) in the xy plane, and let ϕ be the azimuthal angle around the x axis. Each shell is a solid obtained from an annulus $(r, r + \Delta r)$ in the xy plane by revolving it about the x axis. Each ring intersects the xy plane in a ‘polar rectangle,’ with radius in $(r, r + \Delta r)$ and polar angle in $(\theta, \theta + \Delta\theta)$. In Cartesian coordinates one corner of this polar rectangle is at $(x, y) = (r \cos \theta, r \sin \theta)$.

The approximate volume of one chunk is

$$\Delta V = (\Delta r)(r \Delta\theta)(r \sin \theta \Delta\phi).$$

since the distance of the ring from the x axis is about $y = r \sin \theta$. If the mass of the whole ball is M and its volume V , then the mass of the chunk is $\Delta M = M\Delta V/V$, and the magnitude of the force between the chunk and the point mass is

$$|\Delta F| = \frac{Gm\Delta M}{D^2},$$

where D is the distance, given by the Pythagorean theorem as

$$D = ((R - r \cos \theta)^2 + (r \sin \theta)^2)^{1/2}.$$

The force is a vector quantity, however, and only the component along the x axis matters. To get this component, multiply by the cosine of the angle ψ between the x axis and a line through the point mass and the chunk. This is the same for each point around the ring as ϕ varies, and we find

$$\cos \psi = \frac{R - r \cos \theta}{D}.$$

The total force is the sum of x components of force for each chunk:

$$\begin{aligned} F &= \sum_{\text{chunks}} \Delta F_x = \sum \frac{Gm\Delta M}{D^2} \cos \psi = \frac{GmM}{V} \sum \frac{\Delta V}{D^2} \cos \psi \\ &= \frac{GmM}{V} \sum \frac{(\Delta r)(r \Delta \theta)(r \sin \theta \Delta \phi)(R - r \cos \theta)}{D^3} \end{aligned}$$

Leaving off GmM/V , the sum approximates the triple integral

$$\int_{r=0}^{r=A} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{dr(r d\theta)(r \sin \theta d\phi)(R - r \cos \theta)}{((R - r \cos \theta)^2 + (r \sin \theta)^2)^{3/2}}.$$

Since $\int_0^{2\pi} d\phi = 2\pi$, this triple integral equals $\int_0^A I(r) dr$, where

$$I(r) = \int_0^\pi \frac{(2\pi r \sin \theta)(R - r \cos \theta)r d\theta}{((R - r \cos \theta)^2 + (r \sin \theta)^2)^{3/2}}.$$

We evaluate this using two substitutions: First, $x = r \cos \theta$, $dx = -r \sin \theta d\theta$. After switching endpoints of integration and using $r^2 \sin^2 \theta = r^2 - x^2$, we get

$$I(r) = 2\pi r \int_{-r}^r ((R - x)^2 + r^2 - x^2)^{-3/2} (R - x)(+dx)$$

Next, substitute $s = D$, so

$$\begin{aligned} s^2 &= (R - x)^2 + r^2 - x^2 = r^2 + R^2 - 2Rx, \quad 2s ds = -2R dx, \\ R - x &= R + \frac{s^2 - r^2 - R^2}{2R} = \frac{s^2 + R^2 - r^2}{2R}. \end{aligned}$$

We get $s = R + r$ for $x = -r$, $s = R - r$ for $x = r$, hence switching endpoints,

$$\begin{aligned} I(r) &= 2\pi r \int_{R-r}^{R+r} s^{-3} \left(\frac{s^2 + R^2 - r^2}{2R} \right) \frac{s}{R} ds \\ &= \frac{2\pi r}{2R^2} \int_{R-r}^{R+r} (1 + s^{-2}(R^2 - r^2)) ds \\ &= \frac{\pi r}{R^2} \left[2r - \left(\frac{1}{R+r} - \frac{1}{R-r} \right) (R^2 - r^2) \right] = \frac{\pi r}{r^2} (4r) = \frac{4\pi r^2}{R^2}. \end{aligned}$$

Finally we can evaluate

$$\int_0^A I(r) dr = \int_0^A \frac{4\pi r^2}{R^2} dr = \frac{1}{R^2} \cdot \frac{4\pi A^3}{3} = \frac{V}{R^2},$$

and get the total horizontal force:

$$F = \frac{GmM}{V} \frac{V}{R^2} = \frac{GmM}{R^2}.$$