Uniformly continuous functions

Let f be a function defined on a closed interval $[a, b]$. Compare two statements:

(i) $\forall y \in [a, b]\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in [a, b]\ |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$.

(ii) $\forall \varepsilon > 0\, \exists \delta > 0\, \forall y \in [a, b]\, \forall x \in [a, b]\ |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$.

Property (i) says that f is continuous at y for every point $y \in [a, b]$. Property (ii) is in principle stronger, *because the choice of δ does not depend upon y*:

Definition. We say that a function f is *uniformly continuous* on a set $I \subseteq \mathbb{R}$ if and only if

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in I, |x - y| < \delta \to |f(x) - f(y)| < \varepsilon.$$

Counterexample. x^2 is not uniformly continuous on $I = \mathbb{R}$.

Proof. Let $\varepsilon = 1$. Let $\delta > 0$ be arbitrary. For $n \in \mathbb{N}$ let $x_n = \sqrt{n+1}$, $y_n = \sqrt{n}$. Then $x_n^2 - y_n^2 = 1$, but

$$0 < x_n - y_n = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \le \frac{1}{\sqrt{n} + \sqrt{n}} < \delta$$

if $n > (1/2\delta)^2$. Thus $\exists \varepsilon > 0\, \forall \delta > 0\ \exists x, y\ |x - y| < \delta$ and $|x^2 - y^2| \ge \varepsilon$.

Theorem. If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof: 1. Suppose not. Then $\exists \varepsilon_1 > 0$ such that $\forall \delta > 0\ \exists x, y \in [a, b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \ge \varepsilon_1$. Fix this ε_1 and apply the statement to $\delta = 1/n$ for $n = 1, 2, \ldots$ Thus, for every positive integer n there exist $y_n, x_n \in [a, b]$ such that

$$0 < x_n - y_n < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \ge \varepsilon_1.$$

This gives us two lists of numbers $y_1, y_2, \ldots$ and $x_1, x_2, \ldots$ in $[a, b]$.

2. Do bisection as follows: Let $c = (a+b)/2$ and let $I[a, c] = \{n \in \mathbb{P} \mid y_n \in [a, c]\}$. This set is either finite or infinite. Let

$$[a_1, b_1] = \begin{cases} [a, c] & \text{if } I[a, c] \text{ is infinite,} \\ [c, b] & \text{otherwise.} \end{cases}$$

In the second case, $I[c, b]$ must be infinite. By induction, we can construct intervals $[a_m, b_m]$ for $m = 1, 2, \ldots$ such that (i) $b_m - a_m = (b - a)/2^m$, (ii) $a_{m+1} \ge a_m$ and $b_{m+1} \le b_m$, and (iii) $\{n \in \mathbb{P} \mid y_n \in [a_m, b_m]\}$ is infinite.

3. Let $p = \sup\{a_m \mid m = 1, 2, \ldots\}$ and let $\varepsilon_2 = \varepsilon_1/2$. Because f is continuous at p, there exists $\delta_p > 0$ such that $\forall x \in [a, b], |x - p| < \delta \to |f(x) - f(p)| < \varepsilon_2$. Consequently, $\forall x, y \in [a, b]$, $x, y \in N(p, \delta_p)$ implies

$$|f(x) - f(y)| \le |f(x) - f(p)| + |f(p) - f(y)| < \varepsilon_2 + \varepsilon_2 = \varepsilon_1.$$

4. Since $\delta_p > 0$ there exists m such that $(b - a)/2^m < \delta_p/2$. Thus $[a_m, b_m] \subseteq N(p, \delta_p/2)$. Since $\{n \mid y_n \in [a_m, b_m]\}$ is infinite, there exists a positive integer n such that $1/n < \delta_p/2$. Therefore for this n,

$$0 < x_n - y_n < \frac{1}{n} < \frac{\delta_p}{2} \quad \text{and} \quad y_n \in [a_m, b_m] \subseteq N(p, \delta_p/2),$$

hence both $x_n, y_n \in N(p, \delta_p)$, because $|x_n - p| \le |x_n - y_n| + |y_n - p| < \delta_p$. But $|f(x_m) - f(y_m)| \ge \varepsilon_1$. This contradicts the conclusion of step 3. Therefore f *is* uniformly continuous on $[a, b]$.