21-131 Analysis I: Practice for Test 4 R. Pego
Test 4: 11:30 Wednesday, December 3, 2008

The material covered emphasizes sections 3.8 through 4.21 from Apostol, omitting sections 3.12 to 3.16.

1. Find $f'(x)$ for each of the following:

(a) $f(x) = \sin(\sqrt{x^2+1})$ (b) $f(x) = \dfrac{x}{x^3+2}$ (c) $f(x) = \displaystyle\int_1^{x^3} x \sin t \, dt$

2. Find a number $a \in \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_a^x f(t)\,dt = \sin^2 x - 1 \quad \text{for all } x.$$

3. Suppose that on $[a, b]$, f has a relative maximum at a. If $f'(a)$ exists, prove that $f'(a) \le 0$.

4. Find an expression for $\dfrac{dy}{dx}$ in terms of x and y if

$$xy^3 + x^3 y = 10.$$

5. Show that the equation $2x + \sin x + 1 = 0$ has *exactly one* real root. (Hint: For uniqueness, use the Mean Value Theorem.)

6. (i) Suppose that f is continuous at every point in $[a, b]$ and $f(x) > 0$ for all $x \in [a, b]$. Prove that $\{1/f(x) \mid x \in [a, b]\}$ is bounded.
(ii) Give an example of a function f continuous at every point in $(0, 1]$ such that $f(x) > 0$ for all $x \in (0, 1]$ and $\{1/f(x) \mid x \in (0, 1]\}$ is not bounded.

7. Complete the following proof that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$: Let $\varepsilon > 0$ be arbitrary. Let $\delta = \varepsilon^2$. Suppose x, y in $[0, \infty)$ are arbitrary and suppose $0 \le x - y < \delta$. Then there are two cases: $x < \delta$ and $x \ge \delta$. [Rest of proof goes here.]

8. According to a US Geological Survey website, the volcanic dome inside the Mount Saint Helens crater grew rather fast between 2004 and 2008. At one point in time this rate was reported to be 2 cubic meters per second. Assuming the “dome” has the shape of a right circular cone whose height equals its base radius, determine how fast the height was changing with time (in meters per day), if the height was 400 meters.

9. For the function $f : \mathbb{R} \to \mathbb{R}$ defined below, sketch the graph of f, finding all points where f has relative extrema (maxima or minima), and identifying intervals of convexity and concavity. Justify your answers using derivatives.

$$f(x) = \begin{cases} x^3 - 3x^2 + 2x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Topics: Properties of continuous functions on $[a, b]$ (and their uses): Extreme value theorem, intermediate value theorem (solving equations, bisection), mean value theorem (comparisons and estimates). Uniform continuity. Integrability. Sign preservation.

Derivatives by the definition. Derivatives of indefinite integrals. Derivatives of products, quotients, chains of functions. Applications to relating rates of change, implicit differentiation.

A systematic approach to solving word problems on related rates:

- identify variables,
- draw a figure,
- express given information in formulas,
- identify desired result symbolically,
- identify relevant relations between variable quantities,
- differentiate (using chain rule) to relate rates of change,
- solve for answer.

Mean value theorem for derivatives — hypotheses and conclusions. Consequences: using derivatives to find intervals where f is increasing/decreasing, convex/concave. Finding extrema using derivatives.