21-131 Analysis I: Practice for Test 2 R. Pego

Test 2: 11:30 Friday, October 10, 2008

The material covered by test 2 is Apostol chapter 1 and the first part of chapter 2 up to and including 2.5.

1. Evaluate each of the following:

(a) $\int_1^4 \sqrt{x}\,dx$ (b) $\int_{-1}^2 |x| + |x-1|\,dx$

(c) $\int_0^3 \sqrt{x+1}\,dx$ (d) $\int_1^2 (3x^{1/5} - 5(x+2)^3 + 2x - 1)\,dx$

2. Derive the identity

$$\sin 3x = 4 \sin x \sin(x + \frac{1}{3}\pi) \sin(x + \frac{2}{3}\pi).$$

3. Find the area between the graphs of $f(x) = x$ and $g(x) = |x-1|$ in the strip $-1 \le x \le 2$.

4. Assume that $f\colon [0,1] \to [0,1]$ is increasing and that for every $x \in [0,1]$ there is a unique $y \in [0,1]$ such that $f(y) = x$. Define $g\colon [0,1] \to [0,1]$ by $g(x) = y$ if and only if $x = f(y)$. Given that $\int_0^1 f(y)\,dy = \frac{1}{3}$, determine $\int_0^1 g(x)\,dx$.

5. For $b > 0$ define

$$L(b) = \int_1^b \frac{1}{x}\,dx.$$

Using the dilation and interval addition properties of the integral, show that:

- For all a, $b > 0$, $L(b) = L(ab) - L(a)$.
- For all $b > 0$ and all integers n, $L(b^n) = nL(b)$.

6. Assume that f is integrable on $[a,b]$.

(a) Assume that $g(x) = f(x)$ for all $x \in [a,b]$ *except* at one point z. Show that g is integrable on $[a,b]$ and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

(b) Assume that $g(x) = f(x)$ for all $x \in [a,b]$ *except* at N points z_m, $m = 1, \ldots, N$. Show that g is integrable on $[a,b]$ and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

7. Let $f\colon [0,2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable on $[0,2]$? Why or why not? (Suggestion: Show that if s and t are any step functions such that $s(x) \le f(x) \le t(x)$ for all $x \in [0,2]$, then $\int_0^2 s(x)\,dx \le 0$ and $1 \le \int_0^2 t(x)\,dx$.)

8. Assume that

$$\frac{1}{2n} - \frac{1}{n^2} \le \sum_{k=n}^{2n-1} \frac{1}{k^2}, \qquad \sum_{k=n+1}^{2n} \frac{1}{k^2} \le \frac{1}{2n} + \frac{3}{n^2}.$$

Prove that $x \mapsto 1/x^2$ is integrable on $[1,2]$ and $\int_1^2 (1/x^2)\,dx = 1/2$.