21-131 Assignment 6: due Tuesday October 7

6.1–4. From Apostol page 83, do problems 16, 18, 23, 27.

6.5-7. From Apostol page 94, do problems 4, 7, 19.

6.8. (10 points) Prove that *if f and $\hat{f}$ are bounded non-negative functions that are integrable on $[a, b]$, then $f\hat{f}$ is integrable on $[a, b]$,* by filling in the gaps in the following proof (including unmarked ones if you think there are any).

Proof: Suppose f and $\hat{f}$ are bounded, non-negative and integrable on $[a, b]$. Since f and $\hat{f}$ are bounded, there exist real numbers M and $\hat{M}$ such that

$$0 \le f(x) \le M \quad \text{and} \quad 0 \le \hat{f}(x) \le [\mathbf{GAP}] \quad \text{for all } x \in [a, b].$$

We claim: $\exists C > 0$ $\forall n \in \mathbb{P}$ there exist step functions S_n and T_n such that $S_n(x) \le f(x)\hat{f}(x) \le T_n(x)$ for all $x \in [a, b]$ and

$$0 \le \int_a^b T_n(x) - \int_a^b S_n(x)\,dx \le \frac{C}{n}.$$

It follows from this, because $[\mathbf{GAP}]$, that

$$0 \le \overline{I}(f\hat{f}) - \underline{I}(f\hat{f}) \le \frac{C}{n}$$

for all $n \in \mathbb{P}$, consequently $[\mathbf{GAP}]$ and hence $f\hat{f}$ is integrable by Theorem 1.9.

To prove the claim we argue as follows. Let $n \in \mathbb{P}$ be arbitrary. Since f is integrable, $\overline{I}(f) = \underline{I}(f)$, hence, by Theorem I.32 with $h = 1/n$, there exist step functions s_n, t_n such that

$$0 \le s_n(x) \le f(x) \le t_n(x) \le M \quad \text{for all } [\mathbf{GAP}],$$

and

$$\int_a^b t_n(x)\,dx - \frac{1}{n} \le \ [\mathbf{GAP}] \le \int_a^b s_n(x)\,dx + \frac{1}{n}.$$

Similarly, since $\hat{f}$ is integrable, there exist $[\mathbf{GAP}]$.

Let $S_n(x) = s_n(x)\hat{s}_n(x)$ and $T_n(x) = t_n(x)\hat{t}_n(x)$. Then S_n and T_n are step functions and $0 \le S_n(x) \le f(x)\hat{f}(x) \le T_n(x)$ for all $x \in [a, b]$. Furthermore, for all $x \in [a, b]$ we have

$$\begin{aligned} 0 \le T_n(x) - S_n(x) &= (t_n(x) - s_n(x))\hat{t}_n(x) + s_n(x)(\hat{t}_n(x) - \hat{s}_n(x)) \\ &\le (t_n(x) - s_n(x))\hat{M} + M(\hat{t}_n(x) - \hat{s}_n(x)). \end{aligned}$$

Therefore, by the comparison and linearity properties for step functions,

$$\begin{aligned} \int_a^b (T_n(x) - S_n(x))\,dx &\le \hat{M} \int_a^b (t_n(x) - s_n(x))\,dx + [\mathbf{GAP}] \\ &\le \hat{M}\left(\frac{2}{n}\right) + M\left(\frac{2}{n}\right) = \frac{C}{n} \end{aligned}$$

where $C =[\mathbf{GAP}]$. This proves the claim, and finishes the proof.