

21-131 Assignment 5: due Tuesday September 30

5.1. From Apostol page 71, do problem 12.

5.2. Let $x_k = k/(k+1)$ for $k \in \mathbb{N}$. Then $0 = x_0 < x_1 < \dots$ is an infinite list of real numbers in $[0, 1]$ such that $\sup\{x_k \mid k \in \mathbb{N}\} = 1$. Suppose that for $0 \leq x < 1$ we define

$$f(x) = \begin{cases} 1 & \text{for } x \in [x_0, x_1) \cup [x_2, x_3) \cup [x_4, x_5) \cup \dots, \\ 0 & \text{for } x \in [x_1, x_2) \cup [x_3, x_4) \cup [x_5, x_6) \cup \dots \end{cases}$$

and $f(1) = 1$. Show f is integrable on $[0, 1]$. (Note, f is not a step function on $[0, 1]$ because it is not piecewise constant on subintervals of a finite partition.) Suggestion: For each $n \in \mathbb{P}$, use the partition $\{x_0, x_1, \dots, x_n, 1\}$ to define step functions s_n and t_n satisfying $s_n(x) \leq f(x) \leq t_n(x)$ for all $x \in [0, 1]$ and satisfying

$$\int_0^1 t_n(x) dx < \int_0^1 s_n(x) dx + \frac{1}{n}.$$

5.3. Suppose \mathcal{S} is a set of functions g defined on an interval $[a, b]$, all of which are increasing. Suppose also that $\sup\{g(b) \mid g \in \mathcal{S}\}$ exists.

- (i) Prove that for all $x \in [a, b]$, $\sup\{g(x) \mid g \in \mathcal{S}\}$ exists.
- (ii) For $x \in [a, b]$, let $f(x) = \sup\{g(x) \mid g \in \mathcal{S}\}$. Prove that f is increasing. (Hint: Let $x, y \in [a, b]$ be arbitrary and suppose $x < y$. Try to show that $f(y)$ is an upper bound for $\{g(x) \mid g \in \mathcal{S}\}$.)

5.4. Let r_1, r_2, \dots , be an infinite list of positive rational numbers that contains every positive rational number. (As we discussed, such lists exist!)

Let H be the Heaviside function: $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x \leq 0$.

(Note $0 \leq H(x) \leq 1$ for all x .) For each positive integer n let

$$g_n(x) = \sum_{k=1}^n \left(\frac{1}{2}\right)^k H(x - r_k).$$

We have shown that for all x , $0 \leq g_n(x) \leq 1$. For each $x \in \mathbb{R}$ let

$$f(x) = \sup\{g_n(x) \mid n \in \mathbb{P}\}.$$

- (i) Show that for each n , g_n is an increasing function on \mathbb{R} .
- (ii) By problem 5.3 above, $f(x)$ exists and f is increasing on \mathbb{R} . Let $m \in \mathbb{P}$ be fixed, and suppose $x > r_m$. If $n \geq m$, show that $g_n(x) \geq g_n(r_m) + (1/2)^m$. Then show that

$$f(x) \geq f(r_m) + (1/2)^m.$$

(Note: This is mind-boggling.)

- (iii) Deduce that f is one-to-one on \mathbb{R} . That is, show that for all x and y , if $x \neq y$ then $f(x) \neq f(y)$.

5.5. Suppose S and T are nonempty sets of real numbers such that $\forall y \in S \forall z \in T \quad y \leq z$.

(i) Prove: $\sup S \leq \inf T$.

(ii) Assume further that for each positive integer n , there exist $y_n \in S$ and $z_n \in T$ such that $z_n \leq y_n + \frac{5}{n}$. Prove: $\sup S = \inf T$.

(This problem is 'revise-until': You will be asked to revise until your proofs are flawless.)