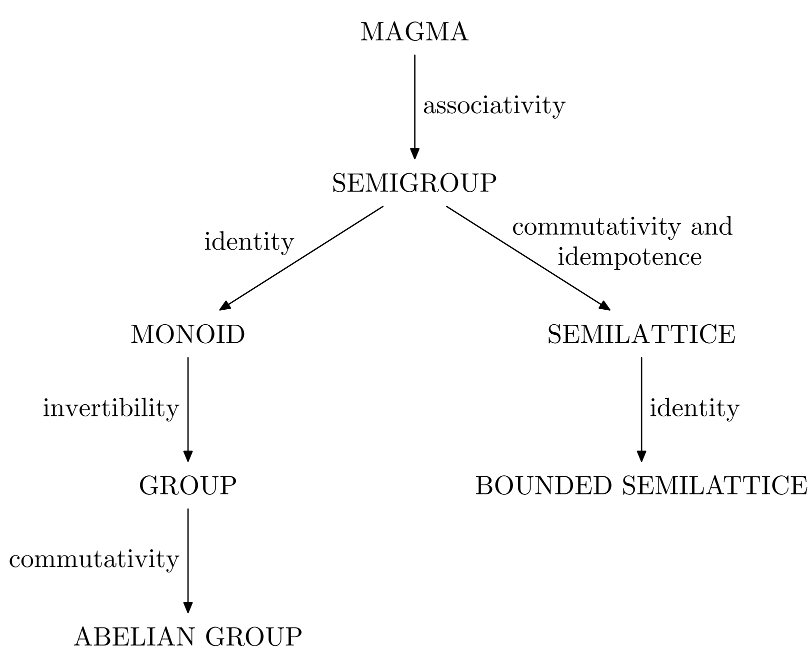
Definitions!

- A **binary operation** on a set S is an operation that takes two elements of S as input and produces one element of S as output.
- A set S is **closed** under an operation if whenever the inputs to the operation come from S, the output of the operation is in S too.
  - [Note that in order for an operation to qualify as a binary operation on a set under the first definition above, the set must be closed under the operation, because the definition of binary operation requires that the output of the operation be in the set. If the set isn't even closed under the operation, then the operation does not qualify as a binary operation on the set, and none of the following definitions apply.]
- A binary operation $\star$ on a set S is **commutative** if $a \star b = b \star a$ for all elements a and b in S.
- A binary operation $\star$ on a set S is **associative** if $(a \star b) \star c = a \star (b \star c)$ for all elements a, b, and c in S.
- Let S be a set with a binary operation $\star$. An element e in S is an **identity element** (or just an **identity**) if $e \star a = a$ and $a \star e = a$ for every element a in S.
- Let S be a set with a binary operation $\star$ and an identity element e.
  - Let a be an element in S. If there exists an element b in S such that $a \star b = e$ and $b \star a = e$, then the element a is **invertible**, and b is an **inverse** of a.
  - If every element a in S is invertible, then the binary operation $\star$ itself is called **invertible**.
- A binary operation $\star$ on a set S is **idempotent** if $a \star a = a$ for every element a in S.

- A set with a binary operation is called a **magma**. [Note that the set must be *closed* under the operation—otherwise the operation wouldn't qualify as a binary operation on the set!]
- There are some special names for magmas that have additional properties.
  - **Semigroup**: associativity.
  - **Monoid**: associativity and identity.
  - **Group**: associativity, identity, and invertibility.
  - **Abelian group**: associativity, identity, invertibility, and commutativity.
  - **Semilattice**: associativity, commutativity, and idempotence.
  - **Bounded semilattice**: associativity, commutativity, idempotence, and identity.



1. The set of positive integers under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

2. The set of nonnegative integers under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

3. The set of all integers under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

4. The set of positive integers under the operation of subtraction.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

5. The set of all integers under the operation of subtraction.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

6. The set of integers under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

7. The set of rational numbers under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

8. The set of integers under the operation of division.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

9. The set of nonzero rational numbers under the operation of division.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

10. The set of positive integers under the operation of *integer division*: division where the remainder is thrown away. For example, under integer division, $38 \div 5 = 7$, because the remainder of 3 is thrown away.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

11. The set of positive integers under the operation of exponentiation.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

12. The set of real numbers under the operation of exponentiation.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

13. The set of rational numbers whose denominators are 1 or 2, under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

14. The set of rational numbers whose denominators are 1, 2, or 3, under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

15. The set of real numbers in the interval $[0, 1]$, under the operation $\leftthreetimes$ defined by $a \leftthreetimes b = \frac{a+b}{2}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

16. The set of all real numbers under the operation $\rightthreetimes$ defined by $a \rightthreetimes b = \frac{a+2b}{3}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

17. The set of rational numbers under the operation $\diamond$ defined by $a \diamond b = \frac{ab}{a+b}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

18. The set of *positive* rational numbers under the operation $\diamond$ defined above.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

19. The set of rational numbers under the operation $\boxplus$ defined by $a \boxplus b = ab + 1$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

20. The set of positive integers under the operation $\circledast$ defined by $a \circledast b = 2^{ab}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

21. The set of real numbers under the operation $\vee$ defined by $a \vee b = \max\{a, b\}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

22. The set of real numbers under the operation $\wedge$ defined by $a \wedge b = \min\{a, b\}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

23. The set of positive integers under the operation $\blacktriangle$ defined by $a \blacktriangle b = \gcd(a, b)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

24. The set of positive integers under the operation $\blacktriangledown$ defined by $a \blacktriangledown b = \mathrm{lcm}(a, b)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

25. The set $\{0, 1, 2, 3, 4\}$ under the operation of addition modulo 5, written $\oplus_5$. Addition modulo 5 is done by adding the two numbers together and then taking the remainder when the sum is divided by 5. For example, $2 \oplus_5 4 = 1$, because $2 + 4 = 6$, and the remainder when 6 is divided by 5 is 1.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

26. The set $\{0, 1, 2, 3, 4\}$ under the operation of multiplication modulo 5, written $\otimes_5$. Multiplication modulo 5 is done by multiplying the two numbers together and then taking the remainder when the product is divided by 5. For example, $3 \otimes_5 4 = 2$, because $3 \times 4 = 12$, and the remainder when 12 is divided by 5 is 2.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

27. The set $\{1, 2, 3, 4\}$ under the operation of multiplication modulo 5.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

28. The set $\{1, 2, 3, 4, 5\}$ under the operation of multiplication modulo 6, written $\otimes_6$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

29. The set of integers under the operation $\triangleleft$ defined by $a \triangleleft b = a$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

30. The set of integers under the operation $\blacktriangleleft$ defined by $a \blacktriangleleft b = -a$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

31. The set of real numbers under the operation ¶ defined by $a \P b =$ the least integer that is greater than $a + b$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

32. The set of states of the U.S., under the operation ♣ defined by $a \clubsuit b =$ Kentucky.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

33. The set of all *sets* of integers, under the operation of union.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

34. The set of all *sets* of integers, under the operation of intersection.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

35. The set of positive integers under the operation $\ltimes$ defined by $a \ltimes b =$ the number you get by writing a down b times. For example, $1702 \ltimes 3 = 170217021702$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

36. The set of nonnegative integers under the operation $\curvearrowright$ defined by $a \curvearrowright b =$ the number you get by doing "move the first digit of a to the end" b times. For example, $12345 \curvearrowright 1 = 23451$, $67890 \curvearrowright 2 = 89067$, $203 \curvearrowright 1 = 32$ (why?), and $149283317 \curvearrowright 57 = 283317149$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

37. The set of Boolean truth values (T and F, "true" and "false") under the operation $\wedge$ defined by $\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$, $\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$, $\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$, and $\mathrm{F} \wedge \mathrm{F} = \mathrm{F}$. (This is the Boolean "AND" operation.)

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

38. The set of Boolean truth values under the operation $\vee$ defined by $\mathrm{T} \vee \mathrm{T} = \mathrm{T}$, $\mathrm{T} \vee \mathrm{F} = \mathrm{T}$, $\mathrm{F} \vee \mathrm{T} = \mathrm{T}$, and $\mathrm{F} \vee \mathrm{F} = \mathrm{F}$. (This is the Boolean "OR" operation.)

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

39. The set of Boolean truth values under the operation $\oplus$ defined by $\mathrm{T} \oplus \mathrm{T} = \mathrm{F}$, $\mathrm{T} \oplus \mathrm{F} = \mathrm{T}$, $\mathrm{F} \oplus \mathrm{T} = \mathrm{T}$, and $\mathrm{F} \oplus \mathrm{F} = \mathrm{F}$. (This is the Boolean "XOR" operation.)

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

40. Rock Paper Scissors: The set $\{r, p, s\}$ under the operation $\bullet$ defined by $r \bullet p = p$ and $p \bullet r = p$ ("paper beats rock"), $p \bullet s = s$ and $s \bullet p = s$ ("scissors beat paper"), $r \bullet s = r$ and $s \bullet r = r$ ("rock beats scissors"), $r \bullet r = r$ ("rock ties with rock"), $p \bullet p = p$ ("paper ties with paper"), and $s \bullet s = s$ ("scissors tie with scissors").

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

41. The set of 2×2 matrices of real numbers, under the operation of matrix addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

42. The set of 2×2 matrices of real numbers, under the operation of matrix multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

43. The set of all finite strings formed from the letters A, B, C, ..., Z, under the operation of string concatenation. (For example, the string "ABC" concatenated with the string "WXYZ" yields the string "ABCWXYZ".)

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

44. The set of polynomial functions in x with integer coefficients, under the operation of function composition, written with the symbol $\circ$, and defined as follows: if f and g are two functions, then $f \circ g$ is the function defined by $f\big(g(x)\big)$. For example, if f is the function defined by $f(x) = 7x^3 + 5x - 12$ and g is the function defined by $g(x) = 4x - 1$, then $f \circ g$ is the function defined by $f\big(g(x)\big)$, which is $7(4x-1)^3 + 5(4x-1) - 12 = 448x^3 - 336x^2 + 104x - 24$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

45. The set of all ordered pairs (x_1, x_2) where x_1 is an integer and x_2 is a real number, under the operation $\ddagger$ defined by $(a_1, a_2) \ddagger (b_1, b_2) = (a_1 + b_1, a_2 \cdot b_2)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

46. The set of permutations of a set of five elements under the operation of composition. Such a permutation can be written as five numbers, like "35142," in which every number from 1 to 5 appears exactly once. The permutation 35142 means that the first element in the output is the third element in the input (that's the 3), and the second element in the output is the fifth element in the input (that's the 5), and the third element in the output is the first element in the input (that's the 1), and so on. For example, applying the permutation 35142 to the input ABCDE gives the output CEADB, and applying the same permutation 35142 to the input BDACE gives the output AEBCD. [Note that a permutation is an *action*—it's a "verb," not a "noun."] The operation of composition means doing one permutation and then another. Composition is written with the symbol $\circ$, and it is done from right to left; so $43125 \circ 35142$ means, "First do the permutation 35142, and then do the permutation 43125 on that output." Applying the composition of permutations $43125 \circ 35142$ to the input ABCDE gives the output DACEB, which is the same output as if you had applied the single permutation 41352; so $43125 \circ 35142 = 41352$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

More problems!

47. The set of even integers under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

48. The set of odd integers under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

49. The set of even integers under the operation of subtraction.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

50. The set of odd integers under the operation of subtraction.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

51. The set of even integers under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

52. The set of odd integers under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

53. The set of positive even integers under the operation of exponentiation.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

54. The set of positive odd integers under the operation of exponentiation.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

55. The set of all rational numbers whose denominators are powers of 2, under the operation of addition.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

56. The set of all nonzero rational numbers whose denominators are powers of 2, under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

57. The set of real numbers under the operation $\odot$ defined by $a \odot b = 7ab$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

58. The set $\{1, 5, 7, 11\}$ under the operation $\otimes_{12}$, multiplication modulo 12. (Remember, multiplication modulo 12 is done by multiplying the two numbers together and then taking the remainder when the product is divided by 12.)

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

59. The set $\{0\}$ under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

60. The set of all real numbers under the operation ⊿ defined by $a \mathbin{⊿} b = \sqrt{a^2 + b^2}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

61. The set of all *nonnegative* real numbers under the operation ⊿ defined above.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

62. The set of all real numbers under the operation ⊿³ defined by $a \mathbin{⊿^3} b = \sqrt[3]{a^3 + b^3}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

63. The set $\mathbb{Z} \cup \{\infty\}$ (that is, the set of integers together with ∞) under the operation $\vee$ defined by $a \vee b = \max\{a, b\}$. Note that $\max\{a, \infty\} = \infty$ for all a in this set, and $\max\{\infty, b\} = \infty$ for all b in this set.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

64. The set $\mathbb{Z} \cup \{-\infty, \infty\}$ (that is, the set of integers together with $-\infty$ and ∞) under the operation $\oplus$ defined by

$$a \oplus b = \begin{cases} 0, & \text{if } a = \infty \text{ and } b = -\infty, \text{ or } a = -\infty \text{ and } b = \infty; \\ \infty, & \text{if } a = \infty \text{ and } b \neq -\infty, \text{ or } b = \infty \text{ and } a \neq -\infty; \\ -\infty, & \text{if } a = -\infty \text{ and } b \neq \infty, \text{ or } b = -\infty \text{ and } a \neq \infty; \\ a + b, & \text{if } a \neq \pm\infty \text{ and } b \neq \pm\infty. \end{cases}$$

For example, $2 \oplus 5 = 7$, $3 \oplus \infty = \infty$, $7412 \oplus -\infty = -\infty$, $-\infty \oplus -\infty = -\infty$, $\infty \oplus \infty = \infty$, and $-\infty \oplus \infty = 0$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

65. The set of positive integers under the operation § defined by $a \,\S\, b = a^b + b^a$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

66. The set of all nonnegative integers that can be expressed as the sum of two perfect squares (i.e., the set $\{0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, \ldots\}$, because, for example, $0 = 0^2 + 0^2$, $5 = 1^2 + 2^2$, $20 = 2^2 + 4^2$, and $25 = 0^2 + 5^2 = 3^2 + 4^2$), under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

67. The set of nonnegative real numbers under the operation ⌭ defined by $a \mathbin{⌭} b = \pi a^2 b$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

68. The set of all positive integers under the operation of tetration, written $\uparrow\uparrow$ and defined by

$$a \uparrow\uparrow b = \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{b \text{ copies of } a}.$$

Note that this "power tower" is evaluated top-down: for example, $7 \uparrow\uparrow 3 = 7^{7^7} = 7^{(7^7)}$, not $(7^7)^7$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

69. The set of perfect squares, $\{0, 1, 4, 9, 16, 25, \ldots\}$, under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

70. The set of Fibonacci numbers under the operation $\blacktriangledown$ defined by $a \blacktriangledown b = \operatorname{lcm}(a, b)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

71. The set of Fibonacci numbers under the operation $\blacktriangle$ defined by $a \blacktriangle b = \gcd(a, b)$.

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☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

72. The set of real numbers of the form $a + b\sqrt{5}$ where a and b are integers, under the operation of multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

73. The set of integers under the operation **?:** defined by

$$a \mathbin{?:} b = \begin{cases} a, & \text{if } a \neq 0; \\ b, & \text{if } a = 0. \end{cases}$$

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

74. The set $\{a, b, c\}$ under the operation $*$ defined by the table below.

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

75. The set $\{0, \frac{1}{2}, 1\}$ under the operation $\rightarrow$ defined by the table below.

$\rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

76. The set $\{x, y, z\}$ under the operation $\leftrightarrow$ defined by the table below.

$\leftrightarrow$	x	y	z
x	x	z	y
y	z	y	x
z	y	x	z

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

77. The set $\{\alpha, \beta, \gamma, \delta\}$ under the operation $\circledcirc$ defined by the table below.

$\circledcirc$	α	β	γ	δ
α	δ	γ	β	α
β	β	β	β	β
γ	γ	γ	γ	γ
δ	α	β	γ	δ

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

78. The subset $\{1, -1, i, -i\}$ of the complex numbers under the operation $\times$ defined by the table below.

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

79. The set of complex numbers under the operation of complex addition, defined by $(a + bi) + (c + di) = (a + c) + (b + d)i$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

80. The set of complex numbers under the operation of complex multiplication, defined by $(a+bi)\cdot(c+di) = (ac - bd) + (bc + ad)i$. [Hint: There is another operation on complex numbers, a *unary* operation, that takes the complex number $a+bi$ as input and produces the complex number $\frac{a}{a^2+b^2} + \left(\frac{-b}{a^2+b^2}\right)i$ as output.]

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

81. The set of *nonzero* complex numbers under the operation of complex multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

82. The set of complex numbers $a+bi$ such that $a^2 + b^2 = 1$, under the operation of complex multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

83. The set of all points inside the unit circle [that is, the set of all points (x, y) whose distance from the origin $(0, 0)$ is less than 1] under the operation $-\!\bullet\!-$ defined by $a -\!\bullet\!- b =$ the midpoint of the line segment joining a and b.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

84. The set of all points *outside* the unit circle under the operation $-\!\bullet\!-$ defined above.

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☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

85. The set of all nonempty, closed, bounded intervals on the real number line (i.e., intervals of the form $[a, b]$ with $a \leq b$), under the operation $\sqcup$ defined by $[a, b] \sqcup [c, d] = [\min\{a, c\}, \max\{b, d\}]$. For example, $[-2.8, 1] \sqcup [\pi, 7] = [-2.8, 7]$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

86. The set of polynomial functions in x with integer coefficients, whose coefficients add up to 0, under the operation of multiplication. For example, $f = 3x^5 - 4x^2 + 1$ is in this set, because $3 - 4 + 1 = 0$, and $g = -17x^3 + 20x^2 - 5x + 2$ is in this set, because $-17 + 20 - 5 + 2 = 0$; and $f \cdot g = (3x^5 - 4x^2 + 1) \times (-17x^3 + 20x^2 - 5x + 2) = -51x^8 + 60x^7 - 15x^6 + 74x^5 - 80x^4 + 3x^3 + 12x^2 - 5x + 2$. [Hint: Think about evaluating one of these polynomials at $x = 1$.]

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

87. The set of all *sets* of integers under the operation of set difference, written $\setminus$ and defined by $A \setminus B =$ the set of all elements of A that are not elements of B. For example, $\{-5, -2, 3, 17, 21\} \setminus \{-2, 0, 14, 17\} = \{-5, 3, 21\}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

88. The set of all *sets* of integers under the operation of symmetric difference, written $\triangle$ and defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$, where $\setminus$ is the operation of set difference defined above. In other words, $A \triangle B$ is the set of all elements of A that are not elements of B, together with all elements of B that are not elements of A. Equivalently, $A \triangle B = (A \cup B) \setminus (A \cap B)$. For example, $\{-5, -2, 3, 17, 21\} \triangle \{-2, 0, 14, 17\} = \{-5, 0, 3, 14, 21\}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

89. The set of all *sets* of integers under the operation of Minkowski addition, written $+$ and defined by $A + B =$ the set of all numbers that you can get by adding one number in A and one number in B. For example, $\{-8, 1, 3\} + \{0, 2, 7\} = \{-8, -6, -1, 1, 3, 5, 8, 10\}$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

90. The set of all finite strings formed from the letters A, B, C, ..., Z, under the operation $\dashv$ defined by $a \dashv b =$ the longest string of letters that appears at the beginning of both a and b. For example, CATFISH $\dashv$ CATAMARAN = CAT and FARMHOUSE $\dashv$ FIREHOUSE = F.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

91. The set of 2×2 matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $ad - bc = 1$, under the operation of matrix multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

92. The set of 2×2 matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that $a + b = 1$ and $c + d = 1$, under the operation of matrix multiplication.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

93. The set of all 8-digit strings of digits 0, 1, 2, ..., 9 (with leading zeroes allowed) under the operation + defined just like ordinary addition, except that leading zeroes are kept in the sum, and if the sum would be a 9-digit number then only the last 8 digits are kept. For example, $02814019 + 03152944 = 05966963$ and $51043819 + 72010038 = 23053857$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

94. The set of all strings, made of the digits 0, 1, 2, ..., 9, that are one-way infinite to the left, under the operation + defined just like ordinary addition, starting at the rightmost digit and proceeding leftwards, with carries. Of course, this process of adding digits (and carrying digits to the left) will require infinitely many steps. For example, $\ldots 31439 + \ldots 52486 = \ldots 83925$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

95. The set of ordered pairs of integers under the operation $\diamond$ defined by $(x_1, y_1) \diamond (x_2, y_2) = (x_1, y_2)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

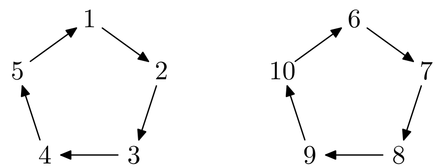
☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

96. The set of ordered pairs of integers under the operation $\bowtie$ defined by $(x_1, y_1) \bowtie (x_2, y_2) = (y_1, x_2)$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

97. The set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ under the operation $\circlearrowright$ defined by $a \circlearrowright b =$ the number you get to by starting at a in the picture below and following the arrows for b steps.



☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

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98. The set of nonnegative integers under the operation $\dotplus$ defined by $a \dotplus b =$ the number that you get by writing a and b in binary and performing binary addition *without carries*, and then converting back to base ten. For example, 12 in binary is 1100, and 42 in binary is 101010; adding those numbers in binary without carries, we get

$$\begin{array}{r} 1100 \\ +\,101010 \\ \hline 100110 \end{array}$$

and 100110 in binary is 38; so $12 \dotplus 42 = 38$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

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99. The set of positive integers under the operation $\textcircled{!}$ defined by the following process: Let a_{first} be the first digit of a, and let a_{last} be the last digit of a. Let b_{first} be the first digit of b, and let b_{last} be the last digit of b. Let a' be the number that you get from a by replacing every occurrence of the digit b_{last} with the digit b_{first}, and let b' be the number that you get from b by replacing every occurrence of the digit a_{last} with the digit a_{first}. Then $a \textcircled{!} b = a' + b'$. For example, $1234 \textcircled{!} 812443 = 1284 + 812113 = 813397$.

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice

100. Quaternions: The set $\{1, -1, i, -i, j, -j, k, -k\}$ under the operation $\times$ defined by the table below.

$\times$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

For example:

$$\begin{array}{lll} i \times i = -1, & i \times j = k, & j \times i = -k, \\ j \times j = -1, & j \times k = i, & k \times j = -i, \\ k \times k = -1, & k \times i = j, & i \times k = -j. \end{array}$$

☐ closed ☐ commutative ☐ associative ☐ identity: ____ ☐ invertible ☐ idempotent

☐ magma ☐ semigroup ☐ monoid ☐ group ☐ abelian group ☐ semilattice ☐ bounded semilattice