PRICING, OPTIMALITY, AND EQUILIBRIUM BASED ON COHERENT RISK MEASURES

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Abstract. The aim of this paper is to apply the theory of coherent risk measures to the problems of finance.

1. First, we study several problems in the theory of coherent risk measures needed for the applications in finance. In particular,

- we give a simple solution to the problem of the *capital allocation* between several units of a firm;
- this result is applied to introduce the notion of *risk contribution* for coherent risk measures;
- furthermore, this result is applied to the problem of finding the optimal structure of a firm consisting of several units.

2. We consider the pricing technique known as *No Good Deals* and establish the fundamental theorem of asset pricing as well as the form of the fair price intervals. We consider two forms of this technique:

- utility-based pricing (employing the assumption that there is no trade with a negative risk);
- RAROC-based pricing (employing the upper limit on a possible RAROC).

Our general model applies to a wide class of coherent risk measures (satisfying only a compactness condition) and to various financial models, including dynamic ones as well as models with an infinite number of assets (in particular, this allows us to consider models with traded derivatives as basic assets, which makes it possible to narrow considerably fair price intervals). Moreover, the proposed approach takes into account such market imperfections as transaction costs, portfolio constraints, liquidity effects, and ambiguity of a historic probability measure.

3. Next we study the optimization problem based on coherent risk measures. This problem is considered in several setups:

- agent-independent optimization (based on RAROC maximization);
- single-agent global optimization;
- single-agent local optimization.

The results are obtained for a general model and are illustrated by a static model with a finite number of assets, where they admit a simple geometric interpretation.

4. Furthermore, the results described above are applied to the optimality pricing. We present several techniques:

- agent-independent optimality pricing;
- single-agent optimality pricing;
- multi-agent optimality pricing.

The results are obtained for a general model and are illustrated by a static model with a finite number of assets, where they admit a simple geometric interpretation.

5. Finally, we consider the equilibrium problem. We establish the equivalence between different definitions and present a criterion for equilibrium.

Furthermore, the equilibrium technique is applied to pricing. Thus, altogether there are at least eight different pricing techniques based on coherent risk measures that are considered in this paper.

Key words and phrases: Ambiguity, capital allocation, coherent risk measure, extreme measure, equilibrium, intersection of risk measures, liquidity, No Good Deals, No Better Choice, portfolio constraints, RAROC, optimality pricing, optimization, risk contribution, risk-neutral measure, Tail V@R, transaction costs, Weighted V@R.

Contents

Introduction

In their landmark papers [1] and [2], Artzner, Delbaen, Eber, and Heath introduced the notion of a *coherent risk measure*. Since these papers, the theory of coherent risk measures has been evolving rapidly. Let us mention, in particular, the papers [15], [21], [29]. Surveys of the modern state of the theory are given in [16], [22; Ch. 4], [33], and [38].

Another interesting topic that has recently emerged in financial mathematics is the theory of *No Good Deals* (*NGD*) pricing. Let us illustrate its idea by an example. Consider a contract that with probability 1/2 yields nothing and with probability 1/2 yields 1000 USD. The No Arbitrage (NA) price interval for this contract is $(0, 1000)$. But if the price of the contract is, for instance, 15 USD, then everyone would be willing to buy it, and the demand would not match the supply. Thus, 15 USD is an unrealistic price because it yields a good deal, i.e. a trade that is attractive to most market participants. The technique of NGD pricing is based on the assumption that good deals do not exist. This technique yields finer price intervals than the NA pricing (the NA price intervals are known to be unacceptably large in incomplete models).

A problem that arises immediately is how to define a good deal. There is no canonical answer, and several approaches have been proposed in the literature. Cochrane and Saá-Requejo [13] defined a good deal as a trade with an unusually high Sharpe ratio, Bernardo and Ledoit [5] based their definition on another gain to loss ratio, while Černý and Hodges [9] proposed a generalization of both definitions (see also the paper [6] by Bjork and Slinko, which extends the results of [13]).

The technique of the NGD pricing can also be motivated as follows. When a trader sells a contract, he would charge for it a price, with which he will be able to superreplicate the contract. In theory the superreplication is typically understood almost surely, but in practice an agent looks for a offsetting position such that the risk of his overall portfolio would stay within the limits prescribed by his management. These considerations lead to the NGD pricing with a good deal defined as a trade with a negative risk. Now, if the risk is measured by V@R, this technique leads to the quantile hedging introduced by Föllmer and Leukert [20]. But instead of V@R, one can take a coherent risk measure. The corresponding pricing technique has already been considered in several papers. Carr, Geman, and Madan [8] (see also the paper [31] by Madan) studied this technique in a probabilistic framework (although they do not use the term "good deal"), while Jaschke and Küchler [25] studied this technique in an abstract framework in the spirit of Harrison and Kreps [23] (see also the paper [37] by Staum, which extends the results of [25]).

As a starting point, we consider the NGD pricing in the probabilistic framework on a fairly general level. Our approach is similar to that of [8] (in fact, several parts of the present paper have been inspired by [8]), but our model is general in the sense that we consider an arbitrary (not only finite) Ω; we consider a general class of coherent risk measures (satisfying only a sort of compactness condition); our approach applies to dynamic models, to models with an infinite number of assets, to models with transaction costs, and to models with convex portfolio constraints. Furthermore, we introduce a variant of the NGD pricing based on the upper limit of a possible Risk-Adjusted Return on Capital (RAROC) defined through coherent risk measures.

Next we pass on from the pricing problem to the optimization problem. Let us mention that some settings of this problem based on coherent risk measures were considered by Barrieu and El Karoui [3], [4] and by Sekine [34]. Here we consider three settings of the optimization problem, which are different from the ones in these papers. The three settings are: agent-independent optimization, single-agent global optimization, and single-agent local optimization.

Then we turn back to the pricing problem and apply the results on optimization to the optimality pricing (three forms of this technique are proposed).

Finally, we study the equilibrium problem and again apply the obtained results to pricing (there are three forms of the equilibrium pricing technique). Our approach has been inspired by the paper [24] by Heath and Ku, and some of the results described below are extensions of the results of [24]. Let us also mention the paper [4] by Barrieu and El Karoui and the paper [27] by Jouini, Schachermayer, and Touzi, which contain a detailed study of two-agent equilibrium and, in particular, the explicit solution of this problem for some classes of risk measures.

1. Coherent risk measures. Before considering pricing, optimality, and equilibrium, we establish in Section 1 several results on coherent risk measures that are needed for applications in financial mathematics. These results are of independent interest.

Recall that Artzner, Delbaen, Eber, and Heath [1], [2] defined a coherent risk measure on a finite space Ω as a map $\rho : L^0 \to \mathbb{R}$ (L^0 denotes the space of all functions on Ω) satisfying four axioms: subadditivity, monotonicity, positive homogeneity, and translation invariance. They proved that any such map has the form

$$\rho(X) = -\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}X, \tag{1}$$

where $\mathcal{D}$ is a convex set of probability measures on Ω. Delbaen [15] defined a coherent risk measure on an arbitrary probability space $(\Omega, \mathcal{F}, \mathsf{P})$ as a map $\rho : L^\infty \to \mathbb{R}$ (L^∞ denotes the space of all bounded random variables) satisfying the above four axioms. He proved that if additionally a continuity axiom (called the Fatou property) is imposed, then representation (1) holds with a set $\mathcal{D} \subseteq \mathcal{P}$, where $\mathcal{P}$ denotes the set of all probability measures that are absolutely continuous with respect to P.

In this paper, we take representation (1) as the definition of a coherent risk measure on the space L^0 of all random variables on $(\Omega, \mathcal{F}, \mathsf{P})$. Thus, we start from a set $\mathcal{D} \subseteq \mathcal{P}$ and define the corresponding *coherent utility function* as

$$u(X) := \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_{\mathsf{Q}}X, \tag{2}$$

where the expectation $\mathsf{E}_{\mathsf{Q}}X$ is understood as $\mathsf{E}_{\mathsf{Q}}X^+ - \mathsf{E}_{\mathsf{Q}}X^-$ (here $X^+ = \max\{X, 0\}$, $X^- = \max\{-X, 0\}$) with the convention: $+\infty - \infty = -\infty$ (in what follows, we will always understand the expectation in this way). Thus, $u : L^0 \to [-\infty, +\infty]$. The corresponding coherent risk measure $\rho(X)$ is defined simply as $-u(X)$. (The term "coherent utility function" was introduced by Delbaen [16]; the use of coherent utility functions instead of coherent risk measures enables one to get rid of numerous minus signs.) Note that different sets $\mathcal{D}$ might lead to the same u, but, for a fixed coherent utility function u, there exists the largest set $\mathcal{D}$, for which representation (2) is true (it is given by $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_{\mathsf{Q}}X \geq u(X) \text{ for any } X\}$). We call it the *determining set* of u. As an example, *Tail V@R* is the risk measure with the determining set $\left\{\mathsf{Q} \in \mathcal{P} : \frac{d\mathsf{Q}}{d\mathsf{P}} \leq \lambda^{-1}\right\}$, where $\lambda \in [0, 1]$ is a fixed parameter. Throughout the paper, we consider coherent utility functions on L^0 (taking representation (2) as the definition).

In Subsection 1.3, we introduce the notion of an *extreme measure.* For an element $X \in L^0$ and a coherent utility function u with the determining set $\mathcal{D}$, the set of extreme measures is defined as

$$\mathcal{X}_{\mathcal{D}}(X) := \{\mathsf{Q} \in \mathcal{D} : \mathsf{E}_{\mathsf{Q}}X = u(X)\}.$$

Proposition 1.9 states that, under some natural conditions imposed on X and $\mathcal{D}$, the set $\mathcal{X}_{\mathcal{D}}(X)$ is nonempty. The notion of an extreme measure turns out to be very important as seen from the results described below.

Subsection 1.4 deals with the problem of the *capital allocation* based on coherent risk measures. This problem was formalized in [16; Sect. 9] and can informally be described as follows. There is a firm consisting of d units. The future income of the i-th unit is given by a random variable X^i. How is the utility $u\bigl(\sum_{i=1}^d X^i\bigr)$ (here u is a coherent utility function) of the whole portfolio allocated between these units? We present (under mild conditions) a geometric solution (see Figure 1) as well as a probabilistic solution of this problem. The latter one states that the set of *utility allocations between* $X^1,\dots,X^d$, i.e. the set of solutions of this problem, has the form

$$\biggl\{\mathsf{E}_{\mathsf{Q}}(X^1,\dots,X^d) : \mathsf{Q}\in\mathcal{X}_{\mathcal{D}}\Bigl(\sum_{i=1}^d X^i\Bigr)\biggr\},$$

where $\mathcal{D}$ is the determining set of u. (A *capital allocation* is defined as a utility allocation with the minus sign.)

The obtained result is applied in Subsection 1.5 to define the notion of a *utility contribution.* The utility contribution $u^c(X;Y)$ of X to Y is defined as

$$u^c(X;Y)=\inf\{x^1 : (x^1,x^2) \text{ is a utility allocation between } X, Y-X\}.$$

(The *risk contribution* is defined as the utility contribution with the minus sign.) It follows from the result described above that (under mild assumptions)

$$u^c(X;Y)=\inf_{\mathsf{Q}\in\mathcal{X}_{\mathcal{D}}(Y)}\mathsf{E}_{\mathsf{Q}}X, \tag{3}$$

so that $u^c(\,\cdot\,;Y)$ is a coherent utility function with the determining set $\mathcal{X}_{\mathcal{D}}(Y)$. The meaning of the utility contribution is clarified by Theorem 1.15, which states (under mild assumptions) that

$$u^c(X;Y)=\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(u(Y+\varepsilon X)-u(Y)).$$

2. Pricing through coherent risk measures. We consider in Section 2 two forms of this technique.

Subsection 2.1 deals with the utility-based NGD pricing. This is done within the framework of a general model introduced in [10]. Thus, we are given a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a convex set $A\subseteq L^0$ termed the *set of attainable incomes.* From the financial point of view, this is the set of discounted incomes that can be obtained in the model under consideration. The fact that A need not be a linear space accounts for transaction costs, liquidity effects, and convex portfolio constraints. Furthermore, we are given a coherent utility function u with the determining set $\mathcal{D}$. The *utility-based NGD* condition is defined as follows: there should exist no $X\in A$ such that $u(X)>0$. Next we define a *risk-neutral measure* as a measure $\mathsf{Q}\in\mathcal{P}$ such that $\mathsf{E}_{\mathsf{Q}}X\le 0$ for any $X\in A$. In view of the convention introduced above, this means that $\mathsf{E}_{\mathsf{Q}}X^-\ge\mathsf{E}_{\mathsf{Q}}X^+$, where both sides are allowed to take on the value $+\infty$ (this particular definition of a risk-neutral measure was introduced in [10]). Theorem 2.4 might be called the fundamental theorem of asset pricing. It states (under mild assumptions that are automatically satisfied in natural models) that the NGD is satisfied if and only if $\mathcal{D}\cap\mathcal{R}\neq\emptyset$, where $\mathcal{R}$ denotes the set of risk-neutral measures. As opposed to the NA fundamental theorems of asset pricing (see [10], [17], [18]), here we need not take a closure of A in defining the NGD.

Then we define a *utility-based NGD price* of a contingent claim F as a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGD condition. Corollary 2.6 states (under mild assumptions) that the set of NGD prices has the form $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$.

Let us remark that the problem of finding utility-based NGD prices can equivalently be formulated as the problem of *NGD superreplication*, which consists in finding the value

$$V^*(F) = \inf\{x \in \mathbb{R} : u(X - F + x) \geq 0 \text{ for some } X \in A\}. \tag{4}$$

Indeed, it is easy to see that $I_{NGD}(F)$ is the interval with the endpoints $V_*(F) := -V^*(-F)$ and $V^*(F)$.

In Subsection 2.2, we introduce the RAROC-based NGD pricing. The framework is as follows. We are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a convex set A of attainable incomes, and two L^1-closed convex sets $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$. We call $\mathcal{PD}$ the *profit-determining set*. This is the set of historic measures used by different market participants. A typical example is $\mathcal{PD} = \{\mathsf{P}\}$. The fact that $\mathcal{PD}$ need not be a singleton accounts for the ambiguity of the historic probability measure. We call $\mathcal{RD}$ the *risk-determining set*. This is the set of scenarios determining a coherent risk measure. Thus, the profit of a possible income X is $\inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_\mathsf{Q} X$, while its risk is $-\inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X$. The RAROC of an income X is defined as

$$\text{RAROC}(X) = \frac{\inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_\mathsf{Q} X}{-\inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_\mathsf{Q} X}.$$

We define the *RAROC-based NGD* condition as the absence of $X \in A$ such that $\text{RAROC}(X) > R$. Here R is a fixed strictly positive number (in practice R can be chosen based on macroeconomic considerations and theories like CAPM). Theorem 2.10 might be called the fundamental theorem of asset pricing. It states (under mild assumptions) that the NGD is satisfied if and only if $\bigl(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\bigr) \cap \mathcal{R} \neq \emptyset$.

Then we define a *RAROC-based NGD price* of a contingent claim F as a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{PD}, \mathcal{RD}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the RAROC-based NGD condition. Corollary 2.12 states (under mild assumptions) that the set of NGD prices has the form

$$\Bigl\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \Bigl(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Bigr) \cap \mathcal{R}\Bigr\}.$$

The general model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ includes, as particular cases, most models of pricing theory, including dynamic ones, models with an infinite number of assets, etc. In order to embed a particular model in the general framework, one should specify the set A and find out the structure of risk-neutral measures (typically, the set $\mathcal{R}$ in a particular model admits a simpler description than the general definition given above).

Subsection 2.3 deals with a static model with a finite number of assets. It is shown that, for this model,

$$\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} S_1 = S_0\},$$

where S_n is the vector of the assets' discounted prices at time n. We also provide for this model a simple geometric interpretation of the general results described above (see Figure 2).

In Subsection 2.4, we consider a continuous-time model with an arbitrary (possibly, infinite) number of assets. Thus, we are given a collection $(S_t^i)_{t \in [0,T]}$, $i \in I$ of adapted càdlàg processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$. Here S_t^i means the discounted price of the i-th asset at time t. It is shown that, for this model,

$$\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \{\mathsf{Q} \in \mathcal{P} : \text{for any } i,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}.$$

Subsection 2.5 is related to the version of this model with transaction costs. Thus, we are given two collections S^{ai}, S^{bi}, $i \in I$ of adapted càdlàg processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P})$. Here S_t^{ai} (resp., S_t^{bi}) means the discounted ask (resp., bid) price of the i-th asset at time t. It is shown that, for this model,

$$\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \{\mathsf{Q} \in \mathcal{P} : \text{for any } i, \text{ there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale } M^i \text{ such that } S^{bi} \le M^i \le S^{ai}\}.$$

Next we consider the following problem. Suppose that we are given a model with proportional transaction costs, i.e. $S^{ai} = S^i$, $S^{bi} = (1 - \lambda^i)S^i$, where each S^i is positive and $\lambda^i \in (0,1)$. Denote the interval of NGD prices in this model by $I_\lambda(F)$. Let $(\lambda_n) = (\lambda_n^i; i \in I, n \in \mathbb{N})$ be a sequence such that $\lambda_n^i \xrightarrow[n\to\infty]{} 0$ for any i. Is it true that

$$I_{\lambda_n}(F) \xrightarrow[n\to\infty]{} I_0(F)? \tag{5}$$

For the NA price intervals, the answer to this question is negative. It was proved in [11], [14], [30], and [36] that, for the Black-Scholes model and $F = (S_T - K)^+$,

$$I_\lambda(F) \xrightarrow[\lambda\downarrow 0]{} ((S_0 - K)^+, S_0)$$

(to be more precise, [14], [30], and [36] study only the convergence of the right endpoints of $I_\lambda(F)$). Although the approaches to the NA pricing in the four papers mentioned above are not the same, this result holds true for all of them. This shows that the NA pricing in continuous-time models with transaction costs typically yields trivial price intervals. In the present paper (Theorem 2.17), we show that, for the NGD price intervals, convergence (5) holds true under natural assumptions. The reason for this discrepancy between the NGD and the NA is as follows. The NGD price intervals are based on the set $\mathcal{D} \cap \mathcal{R}$ of pricing kernels, while the NA price intervals are based on the set $\mathcal{R}$ of pricing kernels. The latter set is too large, and that is the reason why (5) does not hold for the NA price intervals.

3. Optimization. We consider in Section 3 the agent-independent optimization problem as well as the single-agent optimization problem. The former one consists in maximizing the RAROC over the set A of attainable incomes.

In Subsection 3.2, this problem is considered for a static model with a finite number of assets. We provide a simple geometric solution (see Figure 3).

In Subsection 3.3, the obtained results are applied to the problem of the optimal structure of the firm, which is as follows. There is a firm consisting of d units. Each unit can produce a profit h^iX^i, where X^i is a fixed random variable and h^i is a constant chosen by the firm's management (so that increasing the business of the i-th unit means increasing h^i). The problem is to find a structure $h^1, \dots, h^d$, for which $\mathrm{RAROC}\big(\sum_{i=1}^d h^iX^i\big)$ attains its maximum. Theorem 3.5 states (under some assumptions) that a structure $(h^1, \dots, h^d)$ is optimal if and only if

$$\mathrm{RAROC}^c\Big(X^1; \sum_{i=1}^d h^iX^i\Big) = \dots = \mathrm{RAROC}^c\Big(X^d; \sum_{i=1}^d h^iX^i\Big),$$

where the *RAROC contribution of X to Y* is defined as

$$\mathrm{RAROC}^c(X; Y) = \frac{\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_\mathsf{Q}X}{-u^c(X; Y)},$$

where u is the coherent utility function with the determining set $\mathcal{RD}$.

The problem of single-agent optimization consists in finding an optimal element of the set $W+A$, where the random variable W means the agent's current endowment. As opposed to agent-independent optimization, it is reasonable here to consider both RAROC maximization and utility maximization. In essence, RAROC maximization can be reduced to utility maximization by the technique of Lagrange multipliers (as described in Subsection 2.2). For this reason, we consider only utility-based maximization. We study this problem in two forms, which we call global optimization and local optimization.

Global optimization is considered in Subsection 3.4 within the framework of a general model $(\Omega,\mathcal{F},\mathsf{P},A)$. The term "global" means that the set A of attainable incomes is a cone (so that it contains elements that are large as compared to W). Theorem 3.7 states (under mild assumptions) that

$$\sup_{X\in A} u(W+X) = \inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E}_{\mathsf{Q}}W,$$

where $\mathcal{D}$ is the determining set of u.

Let us remark that the optimization problem

$$u(W+X) \xrightarrow[X\in A]{} \max$$

coincides with the problem of NGD superreplication (see (5)). Namely, if we put $F=-W$, then $\sup_{X\in A} u(W+X) = -V^*(F)$, where V^* is given by (4); furthermore, $\operatorname{argmax}_{X\in A} u(W+X)$ coincides with the class of superreplicating strategies for F, i.e. with the class of $X\in A$ such that $u(X-F+V^*(F))\ge 0$.

Subsection 3.5 deals with the problem of global optimization for a static model with a finite number of assets. We provide a simple geometric description of an optimal strategy (see Figure 5).

In Subsection 3.6, we study the problem of local optimization within the framework of a general model $(\Omega,\mathcal{F},\mathsf{P},A)$. From the financial point of view, the term "local" has the following sense: we are considering a big agent and a set A of trading opportunities, each of which is small as compared to the current endowment W of the agent. Mathematically, we impose the condition that A is bounded in a certain sense and study the problem of finding the value

$$u_* = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Big[\sup_{X\in A} u(W+\varepsilon X) - u(W)\Big]$$

and an element $X_*\in A$, for which

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[u(W+\varepsilon X_*) - u(W)] = u_*. \tag{6}$$

Theorem 3.10 shows that this problem is equivalent to the problem

$$u^c(X;W) \xrightarrow[X\in A]{} \max. \tag{7}$$

Namely, the theorem states (under mild assumptions) that $u_* = \sup_{X\in A} u^c(X;W)$ and $X_*\in A$ solves (6) if and only if $X_*\in \operatorname{argmax}_{X\in A} u^c(X;W)$. In view of representation (3), this result admits the following interpretation: in evaluating a trading opportunity, a big agent should use the coherent utility function with the determining set $\mathcal{X}_{\mathcal{D}}(W)$.

In view of representation (3), problem (7) is the problem of maximizing a coherent utility function over a set of attainable incomes (clearly, it makes sense only if A is bounded in a certain sense). In Subsection 3.7, we give a geometric solution of the latter problem in a static model with a finite number of assets (see Figure 7).

In Subsection 3.8, we apply the results on single-agent optimization to the study of liquidity effects in pricing. Within the framework of a general model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A)$, we define the *upper utility-based NGD price function* by

$$\overline{P}_F(v) = \sup\{x : \text{the model } (\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A - v(F - x)) \text{ satisfies the NGD}\}, \quad v > 0,$$

where the NGD is understood in the sense of Subsection 2.1 (the results described below admit a straightforward extension to the RAROC-based NGD by the technique of Lagrange multipliers). The *lower utility-based NGD price function* is defined in a similar way. This approach to liquidity effects in pricing has been inspired by [8]. Theorem 3.13 states that $\overline{P}_F$ is increasing and continuous; furthermore,

$$\lim_{v\downarrow 0} \overline{P}_F(v) = \sup_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}} \mathsf{E}_\mathsf{Q} F, \qquad \lim_{v\to\infty} \overline{P}_F(v) = \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} F$$

provided that A is bounded in a certain sense. Similar investigations of the liquidity effects can be performed for other pricing techniques considered in the paper.

4. Optimality pricing. We consider in Section 4 three forms of this technique.

Subsection 4.1 deals with the agent-independent optimality pricing within the framework of the general model of Subsection 2.2. Thus, we are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a profit-determining set $\mathcal{PD}$, a risk-determining set $\mathcal{RD}$, and a convex set $A \subseteq L^0$ of attainable incomes. An *agent-independent No Better Choice* (*NBC*) *price* of a contingent claim F is a real number x such that passing from A to $A + \{h(F-x) : h \in \mathbb{R}\}$ does not increase $\sup_{X\in A} \mathrm{RAROC}(X)$. Theorem 4.2 states (under mild assumptions) that the set of NBC prices has the form

$$\Big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \Big(\frac{1}{1+R_*}\,\mathcal{PD} + \frac{R_*}{1+R_*}\,\mathcal{RD}\Big) \cap \mathcal{R}\Big\},$$

where $R_* = \sup_{X\in A} \mathrm{RAROC}(X)$. Furthermore, if there exists $X_* \in A$ such that $\mathrm{RAROC}(X_*) = R_*$, then the set of NBC prices has a more definite representation

$$\Big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \Big(\frac{1}{1+R_*}\,\mathcal{X}_{\mathcal{PD}}(X_*) + \frac{R_*}{1+R_*}\,\mathcal{X}_{\mathcal{RD}}(X_*)\Big) \cap \mathcal{R}\Big\}.$$

In typical situations, the set of pricing kernels that stands in this formula is a singleton, and then an NBC price is unique.

In Subsection 4.2, we present a simple geometric description of the NBC price intervals in a static model with a finite number of assets (see Figure 10).

Subsection 4.3 deals with the single-agent optimality pricing in a general model. Similarly to the single-agent optimization problem, it admits both the utility-based version and the RAROC-based version. The latter one can be reduced to the former one by the technique of Lagrange multipliers, so we consider only the utility-based single-agent optimality pricing. Thus, we are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a coherent utility function u with the determining set $\mathcal{D}$, a convex set A of attainable incomes, and a random variable W meaning an agent's current endowment. A *single-agent NBC price* of a contingent claim F is defined as a real number x such that $\max_{X\in A,\, h\in\mathbb{R}} u(W+X+h(F-x)) = u(W)$. Theorem 4.8 states (under mild assumptions) that the set of NBC prices has the form

$$\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_\mathcal{D}(W) \cap \mathcal{R}\}. \tag{8}$$

We also provide a geometric representation of this set (see Figure 11).

In many natural situations the calculation of a single-agent NBC price can be simplified by the following observation. Suppose that W is optimal in the sense that $u(W) = \max_{X \in A} u(W+X)$ and the set of single-agent NBC prices of a contingent claim F based on $\mathcal{D}$ and W (with $A = 0$) consists of one point (this situation is typical). Then the set of NBC prices of F based on $\mathcal{D}$, A, and W consists of the same point. So, in this situation A can be eliminated.

Typically, the single-agent NBC pricing technique yields a one-point set of fair prices, but it is peculiar for a particular agent. In order to get an estimate of the overall fair price, one should take several representative agents (for instance, the major banks) and define a fair price as a price, for which there exists no trading opportunity that is attractive to all the agents. This idea was proposed in [8]. In Subsection 4.4, we study this technique assuming that each of the representative agents tries to maximize a coherent utility function. Thus, we are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a collection $u_1, \dots, u_N$ of coherent utility functions with the determining sets $\mathcal{D}_1, \dots, \mathcal{D}_N$, a convex set $A \subseteq L^0$, and a collection of $W_1, \dots, W_N$ of random variables. Here u_n, A, and W_n mean the utility function, the set of attainable incomes, and the current endowment of the n-th agent, respectively. A *multi-agent NBC price* of a contingent claim F is defined as a real number x such that there exists no element $X \in A + \{h(F-x) : h \in \mathbb{R}\}$ with the property: $u_n(W_n + X) > u_n(W_n)$ for any n. Theorem 4.12 states (under mild assumptions) that the set of NBC prices has the form

$$\big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \operatorname{conv}_{n=1}^N (\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R})\big\} = \operatorname{conv}_{n=1}^N \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}\},$$

where "conv" denotes the convex hull. In view of (8), this is the convex hull of the single-agent NBC price intervals corresponding to the agents $1, \dots, N$.

5. Equilibrium. Section 5 deals with the study of equilibrium based on coherent risk measures. One can consider the utility-based equilibrium and the RAROC-based equilibrium. The latter one can in essence be reduced to the former one by the technique of Lagrange multipliers, so that we study only the utility-based equilibrium.

In Subsection 5.1, we consider a complete model. Thus, we are given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, a collection $u_1, \dots, u_N$ of coherent utility functions with the determining sets $\mathcal{D}_1, \dots, \mathcal{D}_N$, a collection $A_1, \dots, A_N$ of convex subsets of L^0, and a collection $W_1, \dots, W_N$ of random variables. Here u_n, A_n, and W_n mean the utility function, the set of attainable incomes, and the current endowment of the n-th agent, respectively. We define the Pareto-type equilibrium and the Arrow-Debreu-type equilibrium. One of equivalent formulations of the Pareto-type equilibrium is as follows: a system is in equilibrium if and only if

$$\sup_{\substack{X_1 \in A_1, \dots, X_N \in A_N, \\ Y_n \in L^0 : \sum_n Y_n = 0}} \sum_{n=1}^N u_n(W_n + X_n + Y_n) = \sum_{n=1}^N u_n(W_n). \tag{9}$$

(The left-hand side of this equality means the maximal overall utility the agents can obtain by using their trading opportunities as well as by exchanging arbitrary contracts). If (9) is not satisfied, then there exist $X_1 \in A_1, \dots, X_N \in A_N$ and $Y_1, \dots, Y_N \in L^0$ with $\sum_n Y_n = 0$ such that $\sum_n u_n(W_n + X_n + Y_n) > \sum_n u_n(W_n)$. Then, by adding to Y_n constants c_n with $\sum_n c_n = 0$, we can get $\widetilde{Y}_1, \dots, \widetilde{Y}_N \in L^0$ with $\sum_n \widetilde{Y}_n = 0$ such that $u_n(W_n + X_n + \widetilde{Y}_n) > u_n(W_n)$ for any n. Thus, the system is not in equilibrium if and only if the agents can exchange some contracts in such a way that the (coherent) utility of each agent is increased.

Theorem 5.3 states that both notions of Pareto-type equilibrium and Arrow-Debreu type equilibrium are equivalent to the following condition: $\bigcap_{n=1}^N(\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n))\neq\emptyset$. Moreover, the set $\mathcal{E}$ of (appropriately defined) *equilibrium price measures* coincides with $\bigcap_{n=1}^N(\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n))$.

Then we define a *complete equilibrium price* of a contingent claim F as a real number x such that the extended model $(\Omega,\mathcal{F},\mathsf{P},\mathcal{D}_n,A_n+\{h(F-x):h\in\mathbb{R}\},W_n)$ is in complete equilibrium. Corollary 5.5 states (under mild assumptions) that the set of equilibrium prices has the form $\{\mathsf{E}_\mathsf{Q}F:\mathsf{Q}\in\mathcal{E}\}$.

Next we consider the following problem. Suppose that the system is not in equilibrium (so that (9) is not satisfied). How far is it from the equilibrium, i.e. what is the difference between the left-hand side and the right-hand side of (9)? Theorem 5.6 states that

$$\sup_{\substack{X_1\in A_1,\dots,X_N\in A_N,\\ Y_n\in L^0:\sum_n Y_n=0}}\sum_{n=1}^N u_n(W_n+X_n+Y_n)=\inf_{\mathsf{Q}\in\bigcap_n(\mathcal{D}_n\cap\mathcal{R}(A_n))}\mathsf{E}_\mathsf{Q}\Big(\sum_{n=1}^N W_n\Big).$$

This statement is closely connected with the operation of *convex convolution* (or the *inf-convolution*) of coherent utility functions that was studied in [4], [16; Sect. 5.2].

Subsection 5.2 deals with an incomplete model. It is the same as the complete one, but instead of exchanging arbitrary contracts, the agents are only allowed to exchange a finite number of contracts whose prices at time 1 are given by random variables $S_1^1,\dots,S_1^d$. We define the Pareto-type equilibrium and the Arrow-Debreu-type equilibrium. Theorem 5.10 states that both these notions are equivalent to the following condition: $\bigcap_{n=1}^N C_n\neq\emptyset$, where $C_n=\{\mathsf{E}_\mathsf{Q}S_1:\mathsf{Q}\in\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n)\}$ (here $\mathcal{R}(A_n)$ denotes the set of risk-neutral measures corresponding to A_n). Moreover, the set of (appropriately defined) *equilibrium price vectors* coincides with $\cap_n C_n$.

For a contingent claim F, we can define an *incomplete equilibrium price* as an equilibrium price vector corresponding to $S_1=F$. It follows from Theorem 5.10 that the set of equilibrium price vectors has the form

$$\bigcap_{n=1}^N\{\mathsf{E}_\mathsf{Q}F:\mathsf{Q}\in\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n)\}.$$

In view of (8), this is the intersection of the single-agent NBC price intervals corresponding to the agents $1,\dots,N$.

Both complete and incomplete equilibrium pricing techniques have the following property: if x is an equilibrium price of a contingent claim F, then at this price there is no demand and no supply for F. But this is an unrealistic situation, so that in practice the equilibrium price intervals will typically be empty. In Subsection 5.3, we introduce the notion of a *demand-supply equilibrium price*. It is a price, at which the total demand for a contingent claim matches the total supply. Theorem 5.14 states (under mild assumptions) that the set of demand-supply equilibrium prices has the form $\operatorname{argmin}_x\sum_{n=1}^N f_n(x)$, where

$$f_n(x)=\inf\{\mathsf{E}_\mathsf{Q}W_n:\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}(A_n),\ \mathsf{E}_\mathsf{Q}F=x\}.$$

Altogether, there are eight pricing techniques based on coherent risk measures considered in the paper. These techniques are summarized in Table 1. This list can be extended by adding the RAROC-based optimality pricing and the RAROC-based equilibrium pricing.

Pricing technique	Inputs	Form of the price interval
Utility-based No Good Deals	$\mathcal{D}$, A	$\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$
RAROC-based No Good Deals	$\mathcal{PD}$, $\mathcal{RD}$, A, R	$\Big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\big) \cap \mathcal{R}\Big\}$
Agent-independent No Better Choice	$\mathcal{PD}$, $\mathcal{RD}$, A	$\Big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \big(\frac{1}{1+R_*}\mathcal{PD} + \frac{R_*}{1+R_*}\mathcal{RD}\big) \cap \mathcal{R}\Big\}$, where $R_* = \sup_{X \in A} \mathrm{RAROC}(X)$
		$\Big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \big(\frac{1}{1+R_*}\mathcal{X}_{\mathcal{PD}}(X_*) + \frac{R_*}{1+R_*}\mathcal{X}_{\mathcal{RD}}(X_*)\big) \cap \mathcal{R}\Big\}$, where $X_* = \operatorname{argmax}_{X \in A} \mathrm{RAROC}(X)$
Single-agent No Better Choice	$\mathcal{D}$, A, W	$\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_\mathcal{D}(W) \cap \mathcal{R}\}$
		$\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_\mathcal{D}(W)\}$ provided that this is a singleton and $u(W) = \max_{X \in A} u(W + X)$
Multi-agent No Better Choice	$\mathcal{D}_1, \dots, \mathcal{D}_N$, A, $W_1, \dots, W_N$	$\big\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathrm{conv}_{n=1}^N(\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R})\big\}$
Complete equilibrium	$\mathcal{D}_1, \dots, \mathcal{D}_N$, $A_1, \dots, A_N$, $W_1, \dots, W_N$	$\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{E}\}$, where $\mathcal{E} = \bigcap_{n=1}^N (\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}(A_n))$
Incomplete equilibrium	$\mathcal{D}_1, \dots, \mathcal{D}_N$, $A_1, \dots, A_N$, $W_1, \dots, W_N$	$\bigcap_{n=1}^N \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}(A_n)\}$
Demand-supply equilibrium	$\mathcal{D}_1, \dots, \mathcal{D}_N$, $A_1, \dots, A_N$, $W_1, \dots, W_N$	$\operatorname{argmin}_x \sum_{n=1}^N f_n(x)$, where $f_n(x) = \inf\{\mathsf{E}_\mathsf{Q} W_n : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}(A_n),\ \mathsf{E}_\mathsf{Q} F = x\}$

Table 1. The form of price intervals provided by various techniques

1 Coherent Risk Measures

1.1 Basic Definitions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space.

Definition 1.1. A *coherent utility function on* L^∞ is a map $u : L^\infty \to \mathbb{R}$ with the properties:

(a) (Superadditivity) $u(X+Y) \geq u(X) + u(Y)$;
(b) (Monotonicity) If $X \leq Y$, then $u(X) \leq u(Y)$;
(c) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \in \mathbb{R}_+$;
(d) (Translation invariance) $u(X+m) = u(X) + m$ for $m \in \mathbb{R}$;
(e) (Fatou property) If $|X_n| \leq 1$, $X_n \xrightarrow{\mathsf{P}} X$, then $u(X) \geq \limsup_n u(X_n)$.

The corresponding *coherent risk measure* is $\rho(X) = -u(X)$.

The theorem below was established in [2] for the case of a finite Ω (in this case, the axiom (e) is not needed) and in [15] for the general case. We denote by $\mathcal{P}$ the set of probability measures on $\mathcal{F}$ that are absolutely continuous with respect to P. Throughout the paper, we identify measures from $\mathcal{P}$ (these are typically denoted by Q) with their densities with respect to P (these are typically denoted by Z).

Theorem 1.2. *A function* u *satisfies conditions* (a)–(e) *if and only if there exists a nonempty set* $\mathcal{D} \subseteq \mathcal{P}$ *such that*

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^\infty. \tag{1.1}$$

Now, we use representation (1.1) to extend coherent utility functions to L^0.

Definition 1.3. A *coherent utility function on* L^0 is a map $u : L^0 \to [-\infty, \infty]$ defined as

$$u(X) = \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0, \tag{1.2}$$

where $\mathcal{D} \subseteq \mathcal{P}$ and $\mathsf{E}_\mathsf{Q} X$ is understood as $\mathsf{E}_\mathsf{Q} X^+ - \mathsf{E}_\mathsf{Q} X^-$ with the convention $\infty - \infty = -\infty$.

Clearly, a set $\mathcal{D}$, for which representations (1.1) and (1.2) are true, is not unique. However, there exists the largest such set given by $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X \geq u(X) \text{ for any } X\}$.

Definition 1.4. We will call the largest set, for which (1.1) (resp., (1.2)) is true, the *determining set of* u.

`Remarks`. `(i)` Clearly, the determining set is convex. For coherent utility functions on L^∞, it is also L^1-closed. However, for coherent utility functions on L^0, it is not necessarily L^1-closed. As an example, take a positive unbounded random variable X_0 such that $\mathsf{P}(X_0 = 0) > 0$ and consider $\mathcal{D}_0 = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 = 1\}$. Clearly, the determining set $\mathcal{D}$ of the coherent utility function $u(X) = \inf_{\mathsf{Q} \in \mathcal{D}_0} \mathsf{E}_\mathsf{Q} X$ belongs to the set $\{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X_0 \geq 1\}$. On the other hand, the L^1-closure of $\mathcal{D}_0$ contains a measure Q_0 concentrated on $\{X_0 = 0\}$.

`(ii)` Let $\mathcal{D}$ be an L^1-closed convex subset of $\mathcal{P}$. Define a coherent utility function u by (1.1) or (1.2). Then $\mathcal{D}$ is the determining set of u. Indeed, assume that the determining set $\mathcal{D}_0$ is greater than $\mathcal{D}$, i.e. there exists $\mathsf{Q}_0 \in \mathcal{D}_0 \setminus \mathcal{D}$. Then, by the Hahn-Banach theorem, we can find $X_0 \in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0} X_0 < \inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} X$, which is a contradiction.

In what follows, we will always consider coherent utility functions on L^0.

Example 1.5. **(i)** *Tail V@R* (the terms *Average V@R*, *Conditional V@R*, and *expected shortfall* are also used) is the risk measure corresponding to the coherent utility function

$$u_\lambda(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X,$$

where $\lambda \in [0,1]$ and

$$\mathcal{D}_\lambda = \Big\{\mathsf{Q} \in \mathcal{P} : \frac{d\mathsf{Q}}{d\mathsf{P}} \le \lambda^{-1}\Big\}. \tag{1.3}$$

In particular, if $\lambda = 0$, then the corresponding coherent utility function has the form $u(X) = \operatorname{essinf}_\omega X(\omega)$. For more information on Tail V@R, see [15; Sect. 6], [16; Sect. 7], [22; Sect. 4.4], [33; Sect. 1.3].

(ii) *Weighted V@R on* L^∞ is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X)\mu(d\lambda), \quad X \in L^\infty,$$

where μ is a probability measure on $[0,1]$.

Weighted V@R on L^0 is the risk measure corresponding to the coherent utility function

$$u_\mu(X) = \inf_{\mathsf{Q}\in\mathcal{D}_\mu} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0, \tag{1.4}$$

where $\mathcal{D}_\mu$ is the determining set of u_μ on L^∞.

Let us remark that, under some regularity conditions on μ, Weighted V@R possesses some nice properties that are not shared by Tail V@R. We consider Weighted V@R as one of the most (or maybe the most) important classes of coherent risk measures. For a detailed study of this risk measure, see [12], which is in fact a continuation of the present paper. □

1.2 Spaces L^1_w and L^1_s

For a subset $\mathcal{D}$ of $\mathcal{P}$, we introduce the *weak* and *strong* L^1-spaces

$$L^1_w(\mathcal{D}) = \Big\{X \in L^0 : \sup_{\mathsf{Q}\in\mathcal{D}} |\mathsf{E}_\mathsf{Q} X| < \infty\Big\},$$
$$L^1_s(\mathcal{D}) = \Big\{X \in L^0 : \lim_{n\to\infty} \sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}|X|I(|X| > n) = 0\Big\}.$$

Clearly, $L^1_s(\mathcal{D}) \subseteq L^1_w(\mathcal{D})$. If $\mathcal{D} = \{\mathsf{Q}\}$ is a singleton, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D}) = L^1(\mathsf{Q})$, which motivates the notation.

In general, $L^1_s(\mathcal{D})$ might be strictly smaller than $L^1_w(\mathcal{D})$. Indeed, let X_0 be a positive unbounded random variable with $\mathsf{P}(X = 0) > 0$ and let $\mathcal{D} = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E}_\mathsf{Q} X = 1\}$. Then $X_0 \in L^1_w(\mathcal{D})$, but $X_0 \notin L^1_s(\mathcal{D})$. (One can also construct a similar counterexample with an L^1-closed set $\mathcal{D}$; see Example 1.10). However, as shown by theorem below, in most natural situations weak and strong L^1-spaces coincide.

Theorem 1.6. **(i)** *If $\mathcal{D}_\lambda$ is the determining set of Tail V@R (see Example 1.5 (i)) with $\lambda \in (0,1]$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

(ii) *If $\mathcal{D}_\mu$ is the determining set of Weighted V@R (see Example 1.5 (ii)) with μ concentrated on $(0,1]$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

(iii) *If all the densities from $\mathcal{D}$ are bounded by a single constant and $\mathsf{P} \in \mathcal{D}$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.*

(iv) *If $\mathcal{D}$ is a convex combination $\sum_{n=1}^N a_n\mathcal{D}_n$, where $\mathcal{D}_1,\dots,\mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n)=L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D})=L^1_s(\mathcal{D})$.*

(v) *If $\mathcal{D}=\operatorname{conv}(\mathcal{D}_1,\dots,\mathcal{D}_N)$, where $\mathcal{D}_1,\dots,\mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n)=L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D})=L^1_s(\mathcal{D})$.*

Lemma 1.7. *If μ is a convex combination $\sum_{n=1}^\infty a_n\delta_{\lambda_n}$, where $\lambda_n\in(0,1]$, then the determining set $\mathcal{D}_\mu$ of Weighted V@R corresponding to μ has the form $\sum_{n=1}^\infty a_n\mathcal{D}_{\lambda_n}$, where $\mathcal{D}_\lambda$ is given by* (1.3).

`Proof`. Denote $\sum_n a_n\mathcal{D}_{\lambda_n}$ by $\mathcal{D}$. For any $\mathsf{Q}=\sum_n a_n\mathsf{Q}_n\in\mathcal{D}$ and any $X\in L^\infty$, we have

$$\mathsf{E}_\mathsf{Q}X=\sum_{n=1}^\infty a_n\mathsf{E}_{\mathsf{Q}_n}X\ge\sum_{n=1}^\infty a_n u_{\lambda_n}(X)=u_\mu(X),$$

so that $\mathcal{D}\subseteq\mathcal{D}_\mu$.

Let us prove the reverse inclusion. Clearly, $\mathcal{D}$ is convex. Let us prove that $\mathcal{D}$ is L^1-closed. Take a sequence $\xi^k=\sum_n a_nZ_n^k\in\mathcal{D}$ that converges in L^1 to a random variable ξ. Applying the Komlos' principle of subsequences (see [28] or [33; Lem 2.10]) to the infinite-dimensional random vectors $Z^k=(Z_1^k,Z_2^k,\dots)$, we get $\widetilde{Z}^k\in\operatorname{conv}(Z^k,Z^{k+1},\dots)$ that converge P-a.s. componentwise to a random vector $\widetilde{Z}^\infty$. Clearly, $\sum_n a_n\widetilde{Z}_n^k\in\mathcal{D}$ and $\widetilde{Z}_n^k\xrightarrow{L^1}\widetilde{Z}_n^\infty$ for any n (note that $|\widetilde{Z}_n^k|\le\lambda_n^{-1}$). Hence, $\xi=\sum_n a_n\widetilde{Z}_n^\infty\in\mathcal{D}$.

Now, assume that there exists $\mathsf{Q}_0\in\mathcal{D}_\mu\setminus\mathcal{D}$. The Hahn-Banach theorem yields the existence of $X_0\in L^\infty$ such that $\mathsf{E}_{\mathsf{Q}_0}X_0<\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_\mathsf{Q}X_0$. Thus, $u_\mu(X_0)<\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_\mathsf{Q}X_0$. On the other hand, it is easy to check that $u_\mu(X)=\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_\mathsf{Q}X$ for any $X\in L^\infty$. The obtained contradiction shows that $\mathcal{D}_\mu\subseteq\mathcal{D}$. □

`Proof of Theorem 1.6.` The only nontrivial statement is (ii). In order to prove it, consider the measures $\widetilde{\mu}=\sum_{k=1}^\infty a_k\delta_{2^{-k}}$, $\bar{\mu}=\sum_{k=1}^\infty a_k\delta_{2^{-k+1}}$, where $a_k=\mu((2^{-k},2^{-k+1}])$. As $u_{\widetilde{\mu}}\le u_\mu\le u_{\bar{\mu}}$, we have $\mathcal{D}_{\widetilde{\mu}}\supseteq\mathcal{D}_\mu\supseteq\mathcal{D}_{\bar{\mu}}$. By Lemma 1.7,

$$\mathcal{D}_{\widetilde{\mu}}=\Bigl\{\sum_{k=1}^\infty a_kZ_k:Z_k\in\mathcal{D}_{2^{-k}}\Bigr\},\qquad\mathcal{D}_{\bar{\mu}}=\Bigl\{\sum_{k=1}^\infty a_kZ_k:Z_k\in\mathcal{D}_{2^{-k+1}}\Bigr\}.$$

Take $X\in L^1_w(\mathcal{D}_\mu)$. Consider $Z_k=2^{k-1}I(X<q_k)+c_kI(X=q_k)$, where q_k is the 2^{-k+1}-quantile of X and c_k is chosen in such a way that $\mathsf{E}_\mathsf{P}Z_k=1$. Then

$$\mathsf{E}_\mathsf{P}Z_kX=\min_{Z\in\mathcal{D}_{2^{-k+1}}}\mathsf{E}_\mathsf{P}ZX.$$

The density $Z_0=\sum_{k=1}^\infty a_kZ_k$ belongs to $\mathcal{D}_{\bar{\mu}}$ and

$$\mathsf{E}_\mathsf{P}Z_0X=\min_{Z\in\mathcal{D}_{\bar{\mu}}}\mathsf{E}_\mathsf{P}ZX.$$

In view of the inclusion $X\in L^1_w(\mathcal{D}_\mu)\subseteq L^1_w(\mathcal{D}_{\bar{\mu}})$, the latter quantity is finite. Thus,

$$\sum_{k=1}^\infty a_k\min_{Z\in\mathcal{D}_{2^{-k+1}}}\mathsf{E}_\mathsf{P}ZX>-\infty,$$

which implies that

$$\sum_{k=1}^\infty a_k\min_{Z\in\mathcal{D}_{2^{-k+1}}}\mathsf{E}_\mathsf{P}Z(-X^-)>-\infty.$$

The same estimate is true for X^+, and therefore,

$$\sum_{k=1}^{\infty} a_k \sup_{Z \in \mathcal{D}_{2^{-k}}} \mathsf{E}_{\mathsf{P}} Z|X| \le 2 \sum_{k=1}^{\infty} a_k \sup_{Z \in \mathcal{D}_{2^{-k+1}}} \mathsf{E}_{\mathsf{P}} Z|X| < \infty. \tag{1.5}$$

It is clear that $X \in L^1$, and thus, for each k,

$$\sup_{Z \in \mathcal{D}_{2^{-k}}} \mathsf{E}_{\mathsf{P}} Z|X| I(|X| > n) \le 2^k \mathsf{E}_{\mathsf{P}} |X| I(|X| > n) \xrightarrow[n \to \infty]{} 0.$$

This, combined with (1.5), yields

$$\begin{aligned}\sup_{Z \in \mathcal{D}_{\mu}} \mathsf{E}_{\mathsf{P}} Z|X| I(|X| > n) &\le \sup_{Z \in \mathcal{D}_{\widetilde{\mu}}} \mathsf{E}_{\mathsf{P}} Z|X| I(|X| > n) \\ &= \sum_{k=1}^{\infty} a_k \sup_{Z \in \mathcal{D}_{2^{-k}}} \mathsf{E}_{\mathsf{P}} Z|X| I(|X| > n) \xrightarrow[n \to \infty]{} 0.\end{aligned}$$

1.3 Extreme Measures

Definition 1.8. Let u be a coherent utility function with the determining set $\mathcal{D}$. Let $X \in L^0$. We will call a measure $\mathsf{Q} \in \mathcal{D}$ an *extreme measure* for X if $\mathsf{E}_{\mathsf{Q}} X = u(X)$.

The set of extreme measures will be denoted by $\mathcal{X}_{\mathcal{D}}(X)$.

Let us recall some general facts related to the *weak topology* on L^1. The weak topology on L^1 is the induced by the duality between L^1 and L^∞ and is usually denoted as $\sigma(L^1, L^\infty)$. The Dunford-Pettis criterion states that a set $\mathcal{D} \subseteq \mathcal{P}$ is weakly compact if and only if it is weakly closed and uniformly integrable. Furthermore, an application of the Hahn-Banach theorem shows that a convex set $\mathcal{D} \subseteq \mathcal{P}$ is weakly closed if and only if it is L^1-closed.

Proposition 1.9. *If $\mathcal{D}$ is weakly compact and $X \in L^1_s(\mathcal{D})$, then $\mathcal{X}_{\mathcal{D}}(X) \neq \emptyset$.*

`Proof`. It is clear that $u(X) > -\infty$. Find a sequence $Z_n \in \mathcal{D}$ such that $\mathsf{E}_{\mathsf{P}} Z_n X \to u(X)$. This sequence has a weak limit point $Z_\infty \in \mathcal{D}$. Clearly, the map $\mathcal{D} \ni Z \mapsto \mathsf{E}_{\mathsf{P}} ZX$ is weakly continuous. Hence, $\mathsf{E}_{\mathsf{P}} Z_\infty X = u(X)$, which means that $Z_\infty \in \mathcal{X}_{\mathcal{D}}(X)$. □

`Remark`. In many typical situations $\mathcal{X}_{\mathcal{D}}(X)$ is a singleton. For instance, this is the case if $\mathcal{D}$ is the determining set of Tail V@R or Weighted V@R (see Example 1.5) and X has a continuous distribution (for details, see [12; Sect. 6]).

The condition that $\mathcal{D}$ should be weakly compact is very weak and is satisfied for the determining sets of most natural coherent risk measures. For example, the determining set $\mathcal{D}_\lambda$ of Tail V@R is weakly compact provided that $\lambda \in (0,1]$. The determining set of Weighted V@R is weakly compact provided that μ is concentrated on $(0,1]$; this follows from the explicit representation of this set provided in [7] (the proof can also be found in [22; Th. 4.73] or [33; Th. 1.53]).

The following example shows that the condition $X \in L^1_s(\mathcal{D})$ in Theorem 1.6 cannot be replaced by the condition $X \in L^1_w(\mathcal{D})$.

Example 1.10. Let $\Omega = [0,1]$ endowed with the Lebesgue measure. Consider $Z_n = \sqrt{n} I_{[0,1/n]} + 1 - n^{-1/2}$, $n \in \mathbb{N}$. Then $Y_n := Z_n - 1 \xrightarrow{L^1} 0$ and therefore, the set

$$\mathcal{D} = \Bigl\{1 + \sum_{n=1}^{\infty} a_n Y_n : a_n \geq 0,\ \sum_{n=1}^{\infty} a_n \leq 1\Bigr\}$$

is convex, L^1-closed, and uniformly integrable. Thus, $\mathcal{D}$ is weakly compact. Now, consider $X(\omega) = \omega^{-1/2}$. Then $\mathsf{E}_{\mathsf{P}} Z_n X = -4 + 2n^{-1/2}$. Thus, $\inf_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} X = -4$, while there exists no $\mathsf{Q} \in \mathcal{D}$ such that $\mathsf{E}_{\mathsf{Q}} X = -4$. □

1.4 Capital Allocation

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the determining set $\mathcal{D}$, and let $X^1, \dots, X^d \in L^1_w(\mathcal{D})$ be the discounted incomes produced by different components of a firm. We will use the notation $X = (X^1, \dots, X^d)$.

Problem (capital allocation): Find $x^1, \dots, x^d \in \mathbb{R}$ such that

$$u\Bigl(\sum_{i=1}^{d} X^i\Bigr) = \sum_{i=1}^{d} x^i, \tag{1.6}$$

$$\forall h^1, \dots, h^d \in \mathbb{R}_+,\quad u\Bigl(\sum_{i=1}^{d} h^i X^i\Bigr) \leq \sum_{i=1}^{d} h^i x^i. \tag{1.7}$$

We will call a solution of this problem a *utility allocation between* $X^1, \dots, X^d$. (A *capital allocation* is defined as a utility allocation with the minus sign.)

From the financial point of view, $-x^i$ is the contribution of the i-th component to the total risk of the firm, or, equivalently, the capital that should be allocated to this component. In order to illustrate the meaning of (1.7), consider the example $h^i = I(i \in J)$, where J is a subset of $\{1, \dots, d\}$. Then (1.7) means that the capital allocated to a part of the firm does not exceed the risk carried by that part.

Let us introduce the notation $C = \mathrm{cl}\{\mathsf{E}_{\mathsf{Q}} X : \mathsf{Q} \in \mathcal{D}\}$, where "cl" denotes the closure. Note that C is convex and compact.

Theorem 1.11. *The set S of utility allocations between $X^1, \dots, X^d$ has the form*

$$S = \Bigl\{x \in C : \sum_{i=1}^{d} x^i = \min_{y \in C} \sum_{i=1}^{d} y^i\Bigr\}. \tag{1.8}$$

Furthermore, for any utility allocation x, we have

$$\forall h^1, \dots, h^d \in \mathbb{R},\quad u\Bigl(\sum_{i=1}^{d} h^i X^i\Bigr) \leq \sum_{i=1}^{d} h^i x^i. \tag{1.9}$$

If moreover $X^1, \dots, X^d \in L^1_s(\mathcal{D})$ and $\mathcal{D}$ is weakly compact, then

$$S = \Bigl\{\mathsf{E}_{\mathsf{Q}} X : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}}\Bigl(\sum_{i=1}^{d} X^i\Bigr)\Bigr\}. \tag{1.10}$$

Proof. (The proof is illustrated by Figure 1.) For $h \in \mathbb{R}^d$, we set

$$L(h) = \Big\{x \in \mathbb{R}^d : \langle h, x\rangle = \min_{y\in C}\langle h, y\rangle\Big\},$$
$$M(h) = \Big\{x \in \mathbb{R}^d : \langle h, x\rangle \ge \min_{y\in C}\langle h, y\rangle\Big\}.$$

Note that

$$u\Big(\sum_{i=1}^d h^i X^i\Big) = \min_{y\in C}\langle h, y\rangle.$$

Hence, the set of points $x \in \mathbb{R}^d$ that satisfy (1.6) is $L(e)$, where $e = (1, \dots, 1)$. The set of points x that satisfy (1.7) is $\bigcap_{h\in\mathbb{R}^d_+} M(h) = C + \mathbb{R}^d_+$. The set of points x that satisfy (1.9) is $\bigcap_{h\in\mathbb{R}^d} M(h) = C$. This proves (1.8) and (1.9). Equality (1.10) follows immediately from (1.8) and the definition of $\mathcal{X}_{\mathcal{D}}$. □

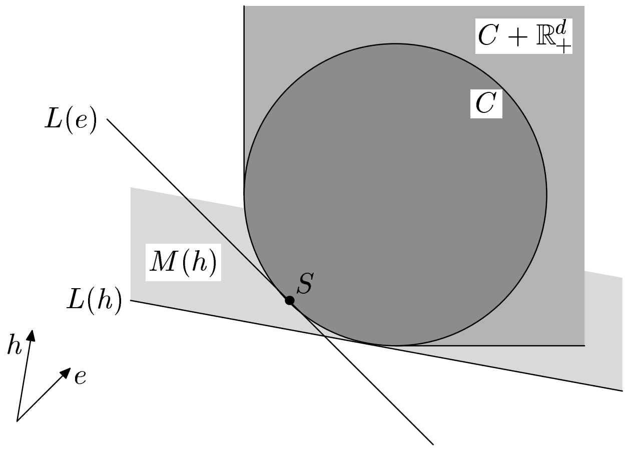


Figure 1. Solution of the capital allocation problem

If C is strictly convex (i.e. its interior is nonempty and its border contains no interval), then a utility allocation is unique. However, in general it is not unique as shown by the example below.

Example 1.12. Let $d = 2$ and $X^2 = -X^1$. Then C is the interval with the endpoints $(u(X^1), -u(X^1))$ and $(-u(-X^1), u(-X^1))$. In this example, $S = C$. □

Let us now find the solution of the capital allocation problem in the Gaussian case.

Example 1.13. Let X have Gaussian distribution with mean a and covariance matrix B. Let u be a *law invariant* coherent utility function, i.e. $u(X)$ depends only on the distribution of X; we also assume that u is finite on Gaussian random variables.

Then there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Bx$. Then the inverse $B^{-1} : L \to L$ is correctly defined. It is easy to see that

$$C = a + \{B^{1/2}x : \|x\| \le \gamma\} = a + \{y \in L : \langle y, B^{-1}y\rangle \le \gamma^2\}.$$

Let $e = (1, \dots, 1)$ and assume first that $Be \ne 0$. In this case the utility allocation x_0 between $X^1, \dots, X^d$ is determined uniquely. In order to find it, note that, for any $y \in L$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\langle x_0 - a + \varepsilon y, B^{-1}(x_0 - a + \varepsilon y)\rangle = 0,$$

we have $\langle e, y\rangle = 0$. This implies that $B^{-1}(x_0 - a) = \alpha \operatorname{pr}_L e$ with some constant α (pr_L denotes the orthogonal projection on L). Thus, $x_0 = a + \alpha Be$. As x_0 should belong to the relative border of C (i.e. the border in the relative topology of $a + L$), we have $\langle x_0 - a, B^{-1}(x_0 - a)\rangle = \gamma^2$, i.e. $\alpha = -\gamma\langle e, Be\rangle^{-1/2}$. As a result, the utility allocation between $X^1, \dots, X^d$ is $a - \gamma\langle e, Be\rangle^{-1/2}Be$.

Assume now that $Be = 0$. This means that e is orthogonal to L, and then the set of utility allocations between $X^1, \dots, X^d$ is C.

Let us remark that in this example the solution of the capital allocation problem depends on u rather weakly, i.e. it depends only on γ. □

1.5 Risk Contribution

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the determining set $\mathcal{D}$, $X \in L^1_w(\mathcal{D})$ be the discounted income produced by a component of some firm, and $Y \in L^1_w(\mathcal{D})$ be the discounted income produced by the whole firm.

Definition 1.14. The *utility contribution of X to Y* is

$$u^c(X;Y) = \inf\{x^1 : (x^1, x^2) \text{ is a utility allocation between } X, Y - X\}.$$

The *risk contribution of X to Y* is defined as $\rho^c(X;Y) = -u^c(X;Y)$.

If $X, Y \in L^1_s(\mathcal{D})$ and $\mathcal{D}$ is weakly compact, then, by Theorem 1.11,

$$u^c(X;Y) = \inf_{\mathsf{Q} \in \mathcal{X}_\mathcal{D}(Y)} \mathsf{E}_\mathsf{Q} X. \tag{1.11}$$

Using this formula, one can extend $u^c(X;Y)$ to all $X \in L^0$.

Remark. If $\mathcal{D}$ is weakly compact and $X^1, \dots, X^d \in L^1_s(\mathcal{D})$ are such that $\mathcal{X}_\mathcal{D}\big(\sum_{i=1}^d X^i\big)$ is a singleton, then (in view of Theorem 1.11) the utility allocation between $X^1, \dots, X^d$ is unique and has the form

$$\bigg(u^c\Big(X^1; \sum_{i=1}^d X^i\Big), \dots, u^c\Big(X^d; \sum_{i=1}^d X^i\Big)\bigg).$$

Theorem 1.15. *If $X, Y \in L^1_s(\mathcal{D})$ and $\mathcal{D}$ is weakly compact, then*

$$u^c(X;Y) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}(u(Y + \varepsilon X) - u(Y)).$$

Proof. Fix $\delta > 0$. The map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} X$ is weakly continuous, and hence, for any $\mathsf{Q} \in \mathcal{X}_\mathcal{D}(Y)$, there exists a neighborhood $V(\mathsf{Q})$ of Q in $\mathcal{D}$ endowed with the weak topology such that

$$\inf_{\mathsf{Q}' \in V(\mathsf{Q})} \mathsf{E}_{\mathsf{Q}'} X \ge \inf_{\mathsf{Q} \in \mathcal{X}_\mathcal{D}(Y)} \mathsf{E}_\mathsf{Q} X - \delta = u^c(X;Y) - \delta.$$

The set $\mathcal{X}_\mathcal{D}(Y)$ is weakly compact being a closed subset of $\mathcal{D}$, so that there exists a neighborhood V of $\mathcal{X}_\mathcal{D}(Y)$ in $\mathcal{D}$ endowed with the weak topology such that

$$\inf_{\mathsf{Q} \in V} \mathsf{E}_\mathsf{Q} X \ge \inf_{\mathsf{Q} \in \mathcal{X}_\mathcal{D}(Y)} \mathsf{E}_\mathsf{Q} X - \delta = u^c(X;Y) - \delta.$$

As $\mathcal{D} \setminus V$ is weakly closed and does not intersect $\mathcal{X}_\mathcal{D}(Y)$, we have

$$\inf_{\mathsf{Q}\in\mathcal{D}\setminus V} \mathsf{E}_\mathsf{Q} Y = u(Y) + a$$

with a strictly positive constant a. It follows from the inequalities

$$\inf_{\mathsf{Q}\in\mathcal{D}\setminus V} \mathsf{E}_\mathsf{Q}(Y+\varepsilon X) \geq u(Y) + a + \varepsilon \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} X,$$

$$\inf_{\mathsf{Q}\in V} \mathsf{E}_\mathsf{Q}(Y+\varepsilon X) \geq \inf_{\mathsf{Q}\in V} \mathsf{E}_\mathsf{Q} Y + \varepsilon \inf_{\mathsf{Q}\in V} \mathsf{E}_\mathsf{Q} X \geq u(Y) + \varepsilon(u^c(X;Y) - \delta)$$

that, for a sufficiently small $\varepsilon > 0$,

$$\inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q}(Y+\varepsilon X) \geq u(Y) + \varepsilon(u^c(X;Y) - \delta).$$

As $\delta > 0$ has been chosen arbitrarily, we get

$$\liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}(u(Y+\varepsilon X) - u(Y)) \geq u^c(X;Y).$$

Combining this with the inequality

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}(u(Y+\varepsilon X) - u(Y)) \leq \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\Big(\inf_{\mathsf{Q}\in\mathcal{X}_\mathcal{D}(Y)} \mathsf{E}_\mathsf{Q}(Y+\varepsilon X) - u(Y)\Big) = u^c(X;Y),$$

we get the desired statement. □

Example 1.16. (i) Let Y be a constant. In this case $\mathcal{X}_\mathcal{D}(Y) = \mathcal{D}$, so that $u^c(X;Y) = u(X)$.

(ii) Let $X = \alpha Y$ with $\alpha \in \mathbb{R}_+$. Then $u^c(X;Y) = \alpha u(Y)$.

(iii) Let X, Y have a jointly Gaussian distribution with mean $(\mathsf{E}X, \mathsf{E}Y)$ and covariance matrix $B = (b_{ij})$. Let u be a law invariant coherent utility function that is finite on Gaussian random variables. Then there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Assume that X and Y are not degenerate and $\mathsf{corr}(X,Y) \neq \pm 1$. It follows from Example 1.13 that

$$\begin{aligned} u^c(X;Y) &= \mathsf{E}X - \gamma \frac{b_{11}+b_{21}}{(b_{11}+b_{21}+b_{12}+b_{22})^{1/2}} \\ &= \mathsf{E}X - \gamma \frac{\mathsf{cov}(X,Y)}{(\mathsf{D}Y)^{1/2}} \\ &= \mathsf{E}X + (u(Y) - \mathsf{E}Y)\frac{\mathsf{cov}(X,Y)}{\mathsf{D}Y}. \end{aligned}$$

In particular, if $\mathsf{E}X = \mathsf{E}Y = 0$, then

$$\frac{u^c(X;Y)}{u(Y)} = \frac{\mathsf{cov}(X,Y)}{\mathsf{D}Y} = \frac{\mathrm{V@RC}(X;Y)}{\mathrm{V@R}(Y)},$$

where V@RC denotes the V@R contribution of X to Y (for the definition, see [32; Sect. 7]). □

2 Good Deals Pricing

2.1 Utility-Based Good Deals Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, and A be a convex subset of L^0. From the financial point of view, A is the set of discounted incomes that can be obtained in the model under consideration (examples are given in Subsections 2.3–2.5). It will be called the *set of attainable incomes*. We will assume that A is $\mathcal{D}$-consistent (see Definition 2.2 below). It is shown in Subsections 2.3–2.5 that this assumption is automatically satisfied for natural models.

Definition 2.1. A *risk-neutral measure* is a measure $\mathsf{Q} \in \mathcal{P}$ such that $\mathsf{E}_\mathsf{Q} X \le 0$ for any $X \in A$ (we use the convention $\mathsf{E}X = \mathsf{E}X^+ - \mathsf{E}X^-$, $\infty - \infty = -\infty$).

The set of risk-neutral measures will be denoted by $\mathcal{R}$ or by $\mathcal{R}(A)$ if there is a risk of ambiguity.

Definition 2.2. We will say that A *is* $\mathcal{D}$*-consistent* if there exists a set $A' \subseteq A \cap L^1_s(\mathcal{D})$ such that $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A')$.

Definition 2.3. A model satisfies the *utility-based NGD* condition if and only if there exists no $X \in A$ such that $u(X) > 0$.

Theorem 2.4. *A model satisfies the NGD condition if and only if* $\mathcal{D} \cap \mathcal{R} \neq \emptyset$.

`Proof`. The "if" part is obvious. Let us prove the "only if" part.

Fix $X_1, \dots, X_M \in A'$. It follows from the weak continuity of the maps $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} X_m$ that the set $C = \{\mathsf{E}_\mathsf{Q}(X_1, \dots, X_M) : \mathsf{Q} \in \mathcal{D}\}$ is compact. Clearly, C is convex. Suppose that $C \cap (-\infty, 0]^M = \emptyset$. Then there exist $h \in \mathbb{R}^M$ and $\varepsilon > 0$ such that $\langle h, x\rangle \ge \varepsilon$ for any $x \in C$ and $\langle h, x\rangle \le 0$ for any $x \in (-\infty, 0]^M$. Hence, $h \in \mathbb{R}^M_+$. Without loss of generality, $\sum_m h_m = 1$. Then $X = \sum_m h_m X_m \in A$ and $\mathsf{E}_\mathsf{Q} X \ge \varepsilon$ for any $\mathsf{Q} \in \mathcal{D}$, so that $u(X) > 0$.

The obtained contradiction shows that, for any $X_1, \dots, X_M \in A'$, the set

$$B(X_1, \dots, X_M) = \{\mathsf{Q} \in \mathcal{D} : \mathsf{E}_\mathsf{Q} X_m \le 0 \text{ for any } m = 1, \dots, M\}$$

is nonempty. As $X_m \in L^1_s(\mathcal{D})$, the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} X_m$ is weakly continuous, and therefore, $B(X_1, \dots, X_M)$ is weakly closed. Furthermore, any finite intersection of sets of this form is nonempty. Consequently, there exists a measure Q that belongs to each B. Then $\mathsf{E}_\mathsf{Q} X \le 0$ for any $X \in A'$, which means that $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}(A')$. As A is $\mathcal{D}$-consistent, $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}$. □

`Remarks`. `(i)` As opposed to the fundamental theorems of asset pricing dealing with the NA condition and its strengthenings (see [10], [17], [18]), here we need not take any closure of A when defining the NGD. Essentially, this is the compactness of $\mathcal{D}$ that yields the fundamental theorem of asset pricing.

`(ii)` If $\mathcal{D} = \mathcal{P}$, then the NGD condition means that there exists no $X \in A$ with $\operatorname{essinf}_\omega X(\omega) > 0$. This is very close to the NA condition. However, in this case $\mathcal{D}$ is not uniformly integrable and Theorem 2.4 might be violated. Indeed, let $A = \{hX : h \in \mathbb{R}\}$, where X has uniform distribution on $[0, 1]$. Then the NGD is satisfied, while $\mathcal{R} = \emptyset$.

Now, let $F \in L^0$ be the discounted payoff of a contingent claim.

Definition 2.5. A *utility-based NGD price* of F is a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGD condition.

The set of NGD prices will be denoted by $I_{NGD}(F)$.

Corollary 2.6. *For* $F \in L^1_s(\mathcal{D})$,

$$I_{NGD}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}.$$

`Proof`. Denote $\{h(F-x) : h \in \mathbb{R}\}$ by $A(x)$. Clearly, $A + A(x)$ is $\mathcal{D}$-consistent (in order to prove this, it is sufficient to consider $A' + A(x)$). It follows from Theorem 2.4 that $x \in I_{NGD}(F)$ if and only if $\mathcal{D} \cap \mathcal{R}(A + A(x)) \neq \emptyset$. It is easy to check that $\mathsf{Q} \in \mathcal{R}(A + A(x))$ if and only if $\mathsf{Q} \in \mathcal{R}$ and $\mathsf{E}_\mathsf{Q} F = x$. This completes the proof. □

`Remark`. As opposed to the NA price intervals, the NGD price intervals are closed (this follows from the weak continuity of the map $\mathcal{D} \cap \mathcal{R} \mapsto \mathsf{E}_\mathsf{Q} F$).

To conclude the subsection, we will discuss the origin of $\mathcal{D}$. First of all, $\mathcal{D}$ might be the determining set of a coherent utility function like Tail V@R or Weighted V@R. The set $\mathcal{D}$ might also correspond to a weighted average or the minimum of several coherent utility functions. It is also possible that $\mathcal{D}$ originates from the classical utility maximization as described by the example below.

Example 2.7. Let $\mathsf{P}_1, \dots, \mathsf{P}_N$ be a family of probability measures, $u_1, \dots, u_N$ be a family of classical utility functions (i.e. smooth concave increasing functions $\mathbb{R} \to \mathbb{R}$), and $W_1, \dots, W_N$ be a family of random variables. From the financial point of view, P_n, u_n, and W_n are the subjective probability, the utility function, and the current endowment of the n-th market participant, respectively. Consider a measure $\mathsf{Q}_n = c_n u'_n(W_n)\mathsf{P}_n$, where c_n is the normalizing constant. Then, for any trading opportunity $X \in L^0$, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} u_n(W_n + \varepsilon X) = \mathsf{E}_{\mathsf{P}_n} u'_n(W_n) X = \mathsf{E}_{\mathsf{Q}_n} c_n^{-1} X \tag{2.1}$$

(we assume that all the expectations exist and integration is interchangeable with differentiation). Thus, an opportunity εX with a small $\varepsilon > 0$ is attractive to the n-th participant if and only if $\mathsf{E}_{\mathsf{Q}_n} F > 0$, so that Q_n might be called the valuation measure of the n-th participant. Take $\mathcal{D} = \operatorname{conv}(\mathsf{Q}_1, \dots, \mathsf{Q}_N)$. Then a good deal is a random variable $X \in A$ such that $\mathsf{E}_{\mathsf{Q}_n} X > 0$ for any n. In view of (2.1), this means that εX with some $\varepsilon > 0$ is attractive to any market participant (this is similar to the notion of a strictly acceptable opportunity introduced in [8]). Thus, in this example the NGD means the absence of a trading opportunity that is attractive to every agent. □

2.2 RAROC-Based Good Deals Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{RD} \subset \mathcal{P}$ be a weakly compact set, $\mathcal{PD}$ be an L^1-closed convex subset of $\mathcal{RD}$, and A be a convex subset of L^0. We will call $\mathcal{PD}$ the *profit-determining set* (from the financial point of view, this is the set of scenarios determining the profit of a position) and $\mathcal{RD}$ the *risk-determining set* (this is the set of scenarios determining the risk of a position). A canonical example is: $\mathcal{PD} = \{\mathsf{P}\}$, while $\mathcal{RD}$ is the determining set of a coherent utility function. We will assume that A is $\mathcal{RD}$-consistent. Finally, we fix a strictly positive number R meaning the upper limit on a possible RAROC.

Definition 2.8. The *Risk-Adjusted Return on Capital* (RAROC) for $X \in L^0$ is defined as

$$\mathrm{RAROC}(X) = \begin{cases} +\infty & \text{if } \mathsf{E_P} X > 0 \text{ and } u(X) \geq 0, \\ \dfrac{\mathsf{E_P} X}{-u(X)} & \text{otherwise} \end{cases}$$

with the convention $\frac{0}{0} = 0$, $\frac{\infty}{\infty} = 0$.

Definition 2.9. A model satisfies the *RAROC-based NGD* condition if and only if there exists no $X \in A$ such that $\mathrm{RAROC}(X) > R$.

Theorem 2.10. *A model satisfies the NGD condition if and only if*

$$\Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \neq \emptyset. \tag{2.2}$$

`Proof`. For any $X \in L^0$,

$$\mathrm{RAROC}(X) > R \iff \inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E_Q} X + R \inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E_Q} X > 0 \iff \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E_Q} X > 0,$$

where $\mathcal{D} = \big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\big)$. Clearly, $\mathcal{D}$ is weakly compact (note that $\mathcal{D} \subseteq \mathcal{RD}$, while $L_s^1(\mathcal{D}) = L_s^1(\mathcal{RD})$) and A is $\mathcal{D}$-consistent. Now, the statement follows from Theorem 2.4. □

Definition 2.11. A *RAROC-based NGD price* of a contingent claim F is a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{PD}, \mathcal{RD}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGD condition.

The set of NGD prices will be denoted by $I_{NGD}(F)$.

Corollary 2.12. *For $F \in L_s^1(\mathcal{D})$,*

$$I_{NGD}(F) = \Big\{\mathsf{E_Q} F : \mathsf{Q} \in \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R}\Big\}.$$

This statement follows from Theorem 2.10.

2.3 Static Model with a Finite Number of Assets

We consider the model of the previous subsection with $A = \{\langle h, S_1 - S_0\rangle : h \in \mathbb{R}^d\}$, where $S_0 \in \mathbb{R}^d$ and $S_1^1, \dots, S_1^d \in L_s^1(\mathcal{RD})$. From the financial point of view, S_n^i is the discounted price of the i-th asset at time n.

Clearly, in this model A is $\mathcal{RD}$-consistent and $\mathcal{RD} \cap \mathcal{R} = \mathcal{RD} \cap \mathcal{M}$, where

$$\mathcal{M} = \{\mathsf{Q} \in \mathcal{P} : \mathsf{E_Q}|S_1| < \infty \text{ and } \mathsf{E_Q} S_1 = S_0\}.$$

`Remark`. We have $\mathcal{M} \subseteq \mathcal{R}$, but the reverse inclusion might be violated. Indeed, let $d = 1$ and let S_1 be such that $\mathsf{E_P} S_1^+ = \mathsf{E_P} S_1^- = \infty$. Then $\mathsf{P} \in \mathcal{R}$, while $\mathsf{P} \notin \mathcal{M}$.

Let us now provide a geometric interpretation of Theorems 2.4 and 2.10. For this, we only assume that $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$ are convex sets and $S_1 \in L_w^1(\mathcal{RD})$.

Let us introduce the notation (see Figure 2)

$$\begin{aligned} B &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{PD}\}, \\ C &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{RD}\}, \\ C_R &= \frac{1}{1+R} B + \frac{R}{1+R} C, \\ D &= \operatorname{conv} \operatorname{supp} \operatorname{Law}_{\mathsf{P}} S_1, \end{aligned}$$

where "supp" denotes the support, and let D° denote the relative interior of D (i.e. the interior in the relative topology of the smallest affine subspace containing D). It is easy to see from the equalities

$$\begin{aligned} \inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E}_{\mathsf{Q}} \langle h, S_1 - S_0 \rangle &= \inf_{x \in B} \langle h, x - S_0 \rangle, \\ \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_{\mathsf{Q}} \langle h, S_1 - S_0 \rangle &= \inf_{x \in C} \langle h, x - S_0 \rangle \end{aligned}$$

that the RAROC-based NGD is satisfied if and only if $S_0 \in C_R$, while the utility-based NGD corresponding to u is satisfied if and only if $S_0 \in C$. Furthermore, it is well known (see [35; Ch. V, § 2e]) that the NA is satisfied if and only if $S_0 \in D^\circ$.

Now, let $F \in L^1_w(\mathcal{RD})$ be the discounted payoff of a contingent claim. Let $\widetilde{B}$, $\widetilde{C}$, $\widetilde{C}_R$, $\widetilde{D}$, and $\widetilde{D}^\circ$ denote the versions of the sets B, C, C_R, D, and D° defined for $\widetilde{S}_1 = (S_1^1, \dots, S_1^d, F)$ instead of S_1. Let $I_{NGD\,(R)}(F)$ denote the RAROC-based NGD price interval, $I_{NGD}(F)$ denote the utility-based NGD price interval (corresponding to u), and $I_{NA}(F)$ denote the NA price interval. Then

$$\begin{aligned} I_{NGD\,(R)}(F) &= \{x : (S_0, x) \in \widetilde{C}_R\}, \\ I_{NGD}(F) &= \{x : (S_0, x) \in \widetilde{C}\}, \\ I_{NA}(F) &= \{x : (S_0, x) \in \widetilde{D}^\circ\}. \end{aligned}$$

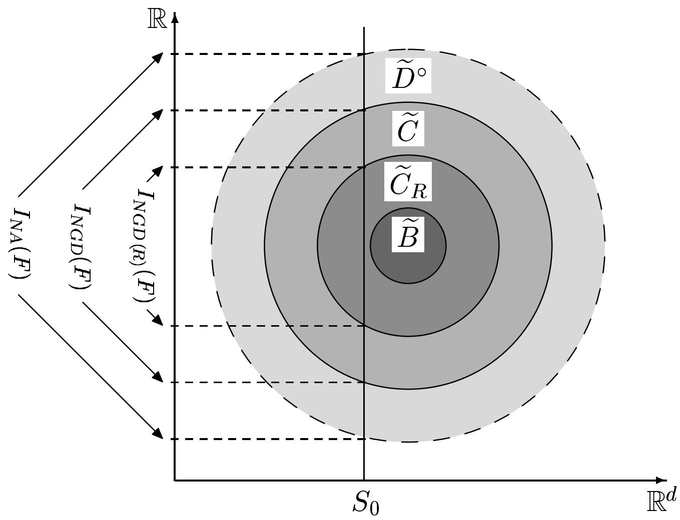


Figure 2. The geometric representation of price intervals provided by various techniques

Example 2.13. Let S_1 have Gaussian distribution with mean a and covariance matrix B. Let $\mathcal{PD} = \{\mathsf{P}\}$ and $\mathcal{RD}$ be the determining set of a law invariant coherent utility function u that is finite on Gaussian random variables. Let F be such that the vector $(S_1^1, \dots, S_1^d, F)$ is Gaussian. Denote $c = \mathsf{cov}(S_1, F)$, $f = \mathsf{E}F$ (we use the vector form of notation).

There exists $b \in \mathbb{R}^d$ such that $Bb = c$. We can write $F = \langle b, S_1 - a\rangle + G + \mathsf{E}F$. Then $\mathsf{E}G = 0$ and $\mathsf{cov}(G, S_1) = 0$, so that G is independent of S_1. Note that

$$d := \mathsf{D}G = \mathsf{D}F - \mathsf{D}\langle b, S_1 - a\rangle = \mathsf{D}F - \langle b, Bb\rangle = \mathsf{D}F - \langle b, c\rangle.$$

Clearly, if $d = 0$, then

$$I_{NGD\,(R)}(F) = I_{NGD}(F) = I_{NA}(F) = \{\langle b, S_0 - a\rangle\}.$$

Let us now assume that $d > 0$.

Obviously, $I_{NA}(F) = \mathbb{R}$.

In order to find $I_{NGD}(F)$, note that $I_{NGD}(F) = \langle b, S_0 - a\rangle + I_{NGD}(G)$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Bx$. Then the inverse $B^{-1} : L \to L$ is correctly defined. As u is law invariant, there exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. From this, it is easy to see that the set $\widetilde{C} := \{\mathsf{E}_\mathsf{Q}(S_1, G) : \mathsf{Q} \in \mathcal{RD}\}$ has the form

$$\widetilde{C} = \{(x, y) : x \in L,\ y \in \mathbb{R} : \langle x, B^{-1}x\rangle + d^{-1}y^2 \le \gamma^2\}.$$

Consequently,

$$I_{NGD}(F) = \big[\langle b, S_0 - a\rangle + f - \alpha, \langle b, S_0 - a\rangle + f + \alpha\big],$$

where $\alpha = (d\gamma^2 - d\langle S_0 - a, B^{-1}(S_0 - a)\rangle)^{1/2}$. (In particular, the NGD is satisfied if and only if $\langle S_0 - a, B^{-1}(S_0 - a)\rangle \le \gamma^2$.)

Similar arguments show that

$$I_{NGD\,(R)}(F) = \big[\langle b, S_0 - a\rangle + f - \alpha(R), \langle b, S_0 - a\rangle + f + \alpha(R)\big], \tag{2.3}$$

where $\alpha(R) = \big(\frac{d\gamma^2 R^2}{1+R^2} - d\langle S_0 - a, B^{-1}(S_0 - a)\rangle\big)^{1/2}$. (In particular, the NGD(R) condition is satisfied if and only if $\langle S_0 - a, B^{-1}(S_0 - a)\rangle \le \frac{\gamma^2 R^2}{1+R^2}$.)

Let us remark that $I_{NGD}(F)$ and $I_{NGD\,(R)}(F)$ depend on u rather weakly, i.e. they depend only on γ. □

2.4 Dynamic Model with an Infinite Number of Assets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is trivial. Let u be a coherent utility function with the weakly compact determining set $\mathcal{D}$. Let (S^i), $i \in I$ be a family of $(\mathcal{F}_t)$-adapted càdlàg processes (the set I is arbitrary). From the financial point of view, S^i is the discounted price process of the i-th asset. We assume that $S_t^i \in L_s^1(\mathcal{D})$ for any $t \in [0, T]$, $i \in I$ (this assumption means that the risk of any simple trade is finite). Define the set of attainable incomes by

$$A = \bigg\{\sum_{n=1}^N \sum_{i\in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \le \dots \le u_N, \text{ are } (\mathcal{F}_t)\text{-stopping times}, \\ H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all } i, \text{ except for a finite set}\bigg\}. \tag{2.4}$$

Lemma 2.14. *We have* $\mathcal{D}\cap\mathcal{R}=\mathcal{D}\cap\mathcal{R}(A')=\mathcal{D}\cap\mathcal{M}$, *where*

$$\begin{aligned}A'&=\{H(S_v^i-S_u^i):u\le v\in[0,T],\ i\in I,\ H\text{ is }\mathcal{F}_u\text{-measurable and bounded}\},\\ \mathcal{M}&=\{\mathsf{Q}\in\mathcal{P}:\text{for any }i\in I,\ S^i\text{ is an }(\mathcal{F}_t,\mathsf{Q})\text{-martingale}\}.\end{aligned}$$

Proof. The inclusions $\mathcal{D}\cap\mathcal{R}\subseteq\mathcal{D}\cap\mathcal{R}(A')\subseteq\mathcal{D}\cap\mathcal{M}$ are clear. So, it is sufficient to prove the inclusion $\mathcal{D}\cap\mathcal{M}\subseteq\mathcal{D}\cap\mathcal{R}$. Let $\mathsf{Q}\in\mathcal{D}\cap\mathcal{M}$. Take $X=\sum_{n=1}^N\sum_{i\in I}H_n^i(S_{u_n}^i-S_{u_{n-1}}^i)\in A$. The process

$$M_k=\sum_{n=1}^k\sum_{i\in I}H_n^i(S_{u_n}^i-S_{u_{n-1}}^i),\quad k=0,\dots,N$$

is an $(\mathcal{F}_{u_k},\mathsf{Q})$-local martingale. Suppose that $\mathsf{E_Q}X^-<\infty$ (otherwise, $\mathsf{E_Q}X=-\infty$). Then M is a martingale (see [35; Ch. II, § 1c]), and hence, $\mathsf{E_Q}X=\mathsf{E_Q}M_N=0$. Thus, in any case, $\mathsf{E_Q}X\le0$, which proves that $\mathsf{Q}\in\mathcal{R}$. □

Example 2.15. Let us consider the Black-Scholes model in the framework of the RAROC-based pricing. Thus, $S_t=S_0e^{\mu t+\sigma B_t}$, where B is a Brownian motion; we are given a weakly compact risk-determining set $\mathcal{RD}$, and we take $\mathcal{PD}=\{\mathsf{P}\}$. Surprisingly enough, in this model $\sup_{X\in A}\text{RAROC}(X)=\infty$. Indeed, the set $\mathcal{M}$ consists of a unique measure Q_0 and $\frac{d\mathsf{Q}_0}{d\mathsf{P}}$ is not bounded away from zero, so that condition (2.2) is violated for any $R>0$.

Let us construct explicitly a sequence $X_n\in A$ with $\text{RAROC}(X_n)\to\infty$. Consider $D_n=\bigl\{\frac{d\mathsf{Q}_0}{d\mathsf{P}}<n^{-1}\bigr\}$ and set $X_n=a_nI(D_n)-I(\Omega\setminus D_n)$, where a_n is chosen in such a way that $\mathsf{E}_{\mathsf{Q}_0}X_n=0$. Then $\mathsf{E_P}X_n\to\infty$, while $\inf_{\mathsf{Q}\in\mathcal{RD}}\mathsf{E_Q}X\ge-1$, so that $\text{RAROC}(X_n)\to\infty$. Actually, $X_n\notin A$, but, for each n, there exists a sequence $(Y_n^m)\in A$ such that $-2\le Y_n^m\le a_n+1$ and $Y_n^m\xrightarrow[m\to\infty]{\mathsf{P}}X_n$ (we leave this to the reader as an exercise). Then $\text{RAROC}(Y_n^m)\xrightarrow[m\to\infty]{}\text{RAROC}(X_n)$, so that $\text{RAROC}\bigl(Y_n^{m(n)}\bigr)\to\infty$ for some subsequence $m(n)$.

This example shows that complete models are typically inconsistent with the RAROC-based NGD pricing. But this technique is primarily aimed at incomplete models because in complete ones the NA price intervals are already exact. □

2.5 Dynamic Model with Transaction Costs

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is trivial and $(\mathcal{F}_t)$ is right-continuous. Let u be a coherent utility function with the weakly compact determining set $\mathcal{D}$. Let S^{ai},S^{bi}, $i\in I$ be two families of $(\mathcal{F}_t)$-adapted càdlàg processes. From the financial point of view, S^{ai} (resp., S^{bi}) is the discounted ask (resp., bid) price process of the i-th asset (so that $S^a\ge S^b$ componentwise). We assume that $S_t^{ai},S_t^{bi}\in L_s^1(\mathcal{D})$ for any $t\in[0,T]$, $i\in I$. Define the set of attainable incomes by

$$\begin{aligned}A=\Biggl\{&\sum_{n=0}^N\sum_{i\in I}\bigl[-H_n^iI(H_n^i>0)S_{u_n}^{ai}-H_n^iI(H_n^i<0)S_{u_n}^{bi}\bigr]:\\ &N\in\mathbb{N},\ u_0\le\dots\le u_N\text{ are }(\mathcal{F}_t)\text{-stopping times, }H_n^i\text{ is }\mathcal{F}_{u_n}\text{-measurable,}\\ &H_n^i=0\text{ for all }i,\text{ except for a finite set, and }\sum_{n=0}^N H_n^i=0\text{ for any }i\Biggr\}.\end{aligned}$$

Here H_n^i means the amount of the i-th asset that is bought at time u_n (so that $\sum_{k=0}^n H_k^i$ is the total amount of the i-th asset held at the time u_n). Note that if there are no transaction costs, i.e. $S^{ai} = S^{bi} = S^i$ for each i, then the set of attainable incomes coincides with the set given by (2.4).

Lemma 2.16. *We have* $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{M}$, *where*

$$\begin{aligned} A' = \{&G(S_v^{bi} - S_u^{ai}) + H(-S_v^{ai} + S_u^{bi}) : i \in I,\ u \le v \text{ are simple } (\mathcal{F}_t)\text{-stopping} \\ &\text{times, } G, H \text{ are positive, bounded, } \mathcal{F}_u\text{-measurable}\}, \\ \mathcal{M} = \{&\mathsf{Q} \in \mathcal{P} : \text{for any } i, \text{ there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale } M^i \\ &\text{such that } S^{bi} \le M^i \le S^{ai}\}. \end{aligned}$$

(*A stopping time is simple if it takes on a finite number of values.*)

Proof. The inclusion $\mathcal{D} \cap \mathcal{R} \subseteq \mathcal{D} \cap \mathcal{R}(A')$ is obvious.

Let us prove the inclusion $\mathcal{D} \cap \mathcal{R}(A') \subseteq \mathcal{D} \cap \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}(A')$. Fix $i \in I$. For any simple stopping times $u \le v$, we have $S_u^{ai}, S_u^{bi}, S_v^{ai}, S_v^{bi} \in L_s^1(\mathcal{D})$ and

$$\mathsf{E}_\mathsf{Q}(S_v^{ai} \mid \mathcal{F}_u) \ge S_u^{bi}, \qquad \mathsf{E}_\mathsf{Q}(S_v^{bi} \mid \mathcal{F}_u) \le S_u^{ai}. \tag{2.5}$$

Consider the Snell envelopes

$$\begin{aligned} X_t &= \operatorname*{esssup}_{\tau \in \mathcal{T}_t} \mathsf{E}_\mathsf{Q}(S_\tau^{bi} \mid \mathcal{F}_t), \quad t \in [0, T], \\ Y_t &= \operatorname*{essinf}_{\tau \in \mathcal{T}_t} \mathsf{E}_\mathsf{Q}(S_\tau^{ai} \mid \mathcal{F}_t), \quad t \in [0, T], \end{aligned}$$

where $\mathcal{T}_t$ denotes the set of simple $(\mathcal{F}_t)$-stopping times such that $\tau \ge t$. (Recall that $\operatorname{esssup}_\alpha \xi_\alpha$ is a random variable ξ such that, for any α, $\xi \ge \xi_\alpha$ a.s. and for any other random variable ξ' with this property, we have $\xi \le \xi'$ a.s.) Then X is an $(\mathcal{F}_t)$-supermartingale, while Y is an $(\mathcal{F}_t, \mathsf{Q})$-submartingale (see [19; Th. 2.12.1]).

Let us prove that, for any $t \in [0, T]$, $X_t \le Y_t$ Q-a.s. Assume that there exists t such that $\mathsf{P}(X_t > Y_t) > 0$. Then there exist $\tau, \sigma \in \mathcal{T}_t$ such that

$$\mathsf{Q}\big(\mathsf{E}_\mathsf{Q}(S_\tau^{bi} \mid \mathcal{F}_t) > \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_t)\big) > 0.$$

This implies that $\mathsf{Q}(\xi > \eta) > 0$, where $\xi = \mathsf{E}_\mathsf{Q}(S_\tau^{bi} \mid \mathcal{F}_{\tau \wedge \sigma})$ and $\eta = \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_{\tau \wedge \sigma})$. Assume first that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \le \sigma\}) > 0$. On the set $\{\tau \le \sigma\}$ we have

$$\xi = S_\tau^{bi} = S_{\tau \wedge \sigma}^{bi}, \qquad \eta = \mathsf{E}_\mathsf{Q}(S_\sigma^{ai} \mid \mathcal{F}_{\tau \wedge \sigma}) = \mathsf{E}_\mathsf{Q}(S_{\tau \vee \sigma}^{ai} \mid \mathcal{F}_{\tau \wedge \sigma}),$$

and we obtain a contradiction with (2.5). In a similar way we get a contradiction if we assume that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \ge \sigma\}) > 0$. As a result, $X_t \le Y_t$ Q-a.s. Now, it follows from [26; Lem. 3] that there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M such that $X \le M \le Y$. As a result, $\mathsf{Q} \in \mathcal{M}$.

Let us prove the inclusion $\mathcal{D} \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{D} \cap \mathcal{M}$, so that, for any i, there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M^i such that $S^{bi} \le M^i \le S^{ai}$. For any

$$X = \sum_{n=0}^N \sum_{i \in I} \big[-H_n^i I(H_n^i > 0) S_{u_n}^{ai} - H_n^i I(H_n^i < 0) S_{u_n}^{bi}\big] \in A,$$

we have

$$X \le \sum_{n=0}^{N}\sum_{i\in I}\big[-H_n^i I(H_n^i>0)M_{u_n}^i - H_n^i I(H_n^i<0)M_{u_n}^i\big] = \sum_{n=1}^{N}\sum_{i\in I}\Big(\sum_{k=0}^{n-1} H_k^i\Big)(M_{u_n}^i - M_{u_{n-1}}^i).$$

Repeating the arguments used in the proof of Lemma 2.14, we get $\mathsf{E}_\mathsf{Q} X \le 0$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Consider now a model with proportional transaction costs, i.e. $S^{ai} = S^i$, $S^{bi} = (1-\lambda^i)S^i$, where each S^i is positive, $\lambda^i \in (0,1)$. Denote the interval of NGD prices in this model by $I_\lambda(F)$. Let $(\lambda_n) = (\lambda_n; i \in I, n \in \mathbb{N})$ be a sequence such that $\lambda_n^i \xrightarrow[n\to\infty]{} 0$ for any i.

Theorem 2.17. *For $F \in L_s^1(\mathcal{D})$, $I_{\lambda_n}(F) \xrightarrow[n\to\infty]{} I_0(F)$ in the sense that the right (resp., left) endpoints of $I_{\lambda_n}(F)$ converge to the right (resp., left) endpoint of $I_0(F)$.*

`Proof`. Let r denote the right endpoint of $I_0(F)$. Suppose that the right endpoints of $I_{\lambda_n}(F)$ do not converge to r. Then there exists $r' > r$ such that, for each n (possibly, after passing on to a subsequence), there exists $\mathsf{Q}_n \in \mathcal{D} \cap \mathcal{R}_{\lambda_n}$ with the property: $\mathsf{E}_\mathsf{Q} F \ge r'$ ($\mathcal{R}_\lambda$ is the set of risk-neutral measures in the model corresponding to λ). The sequence (Q_n) has a weak limit point $\mathsf{Q}_\infty \in \mathcal{D}$. Fix $i \in I$, $u \le v \in [0,T]$, and a positive bounded $\mathcal{F}_u$-measurable function H. For any n, we have $\mathsf{E}_{\mathsf{Q}_n} H((1-\lambda_n^i)S_v^i - S_u^i) \le 0$. As $S_v^i \in L_s^1(\mathcal{D})$, we have $\sup_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} S_v^i < \infty$, and hence, $\limsup_n \mathsf{E}_{\mathsf{Q}_n} H(S_v^i - S_u^i) \le 0$. As the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} H(S_v^i - S_u^i)$ is weakly continuous, we get $\mathsf{E}_{\mathsf{Q}_\infty} H(S_v^i - S_u^i) \le 0$. In a similar way, we prove that $\mathsf{E}_{\mathsf{Q}_\infty} H(-S_v^i + S_u^i) \le 0$. Thus, S^i is an $(\mathcal{F}_t, \mathsf{Q}_\infty)$-martingale, so that $\mathsf{Q}_\infty \in \mathcal{D} \cap \mathcal{R}_0$. As the map $\mathcal{D} \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} F$ is weakly continuous, we should have $\mathsf{E}_{\mathsf{Q}_\infty} F \ge r'$. But this is a contradiction. □

3 Optimization

3.1 Agent-Independent Optimization

We consider the model of Subsection 2.2.

Problem (agent-independent optimization): Find

$$R_* = \sup_{X\in A} \mathrm{RAROC}(X)$$

and

$$X_* = \operatorname*{argmax}_{X\in A} \mathrm{RAROC}(X).$$

The only statement we can make at this level of generality is that

$$R_* = \inf\Big\{R>0 : \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \ne \emptyset\Big\}$$

(this follows from Theorem 2.10). Of course, in general X_* might not exist.

3.2 Static Model with a Finite Number of Assets

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{PD} \subseteq \mathcal{RD} \subseteq \mathcal{P}$ be convex sets, $S_0 \in \mathbb{R}^d$, and $S_1^1, \dots, S_1^d \in L_w^1(\mathcal{RD})$. Let $A = \{\langle h, S_1 - S_0\rangle : h \in H\}$, where $H \subseteq \mathbb{R}^d$ is a convex set such that its cone hull is closed.

Let us introduce the notation (see Figure 3)

$$\begin{aligned} H^* &= \{x \in \mathbb{R}^d : \forall h \in H,\ \langle h, x\rangle \geq 0\},\\ B &= \mathrm{cl}\{\mathsf{E_Q} S_1 : \mathsf{Q} \in \mathcal{PD}\},\\ C &= \mathrm{cl}\{\mathsf{E_Q} S_1 : \mathsf{Q} \in \mathcal{RD}\},\\ D &= C + H^*, \end{aligned} \tag{3.1}$$

and let D° denote the relative interior of D. The sets B and C are convex compacts, while D is convex and closed. Note that, for $h \in H$,

$$\inf_{\mathsf{Q} \in \mathcal{PD}} \mathsf{E_Q} \langle h, S_1 - S_0\rangle = \inf_{x \in B} \langle h, x - S_0\rangle, \tag{3.2}$$

$$\inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E_Q} \langle h, S_1 - S_0\rangle = \inf_{x \in C} \langle h, x - S_0\rangle = \inf_{x \in D} \langle h, x - S_0\rangle. \tag{3.3}$$

We will assume that $S_0 \in D^\circ \setminus B$. This assumption is justified economically. Indeed, if $S_0 \in B$, then, in view of (3.2), $\mathrm{RAROC}(X) = 0$ for any $X \in A$; if $S_0 \notin D^\circ$, then, in view of (3.3), there exists $X \in A$ with $\mathrm{RAROC}(X) = \infty$ (provided that B belongs to the relative interior of C).

For $\lambda > 0$, we denote $B(\lambda) = S_0 - \lambda(B - S_0)$ and set $\lambda_* = \sup\{\lambda > 0 : B(\lambda) \cap D \neq \emptyset\}$,

$$N = \{h \in H : \exists a \in \mathbb{R} : \forall x \in B(\lambda_*), \forall y \in D,\ \langle h, x\rangle \leq a \leq \langle h, y\rangle \text{ and } \forall y \in D^\circ,\ \langle h, y\rangle > a\}.$$

Note that N is nonempty provided that $\lambda_* < \infty$. In the case, where $\lambda_* = \infty$, we set $N = H$.

Theorem 3.1. *We have* $R_* = \lambda_*^{-1}$ *and* $\operatorname{argmax}_{h \in H} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = N$.

`Proof`. We will prove the statement for the case $\lambda_* < \infty$. The proof for the case $\lambda_* = \infty$ is similar. Take $T \in B(\lambda_*) \cap D$ and set $U = S_0 - \lambda_*^{-1}(T - S_0)$.

If $h \in N$, then

$$\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = \frac{\inf_{x \in B} \langle h, x - S_0\rangle}{-\inf_{x \in D} \langle h, x - S_0\rangle} = \frac{\langle h, U - S_0\rangle}{-\langle h, T - S_0\rangle} = \lambda_*^{-1}.$$

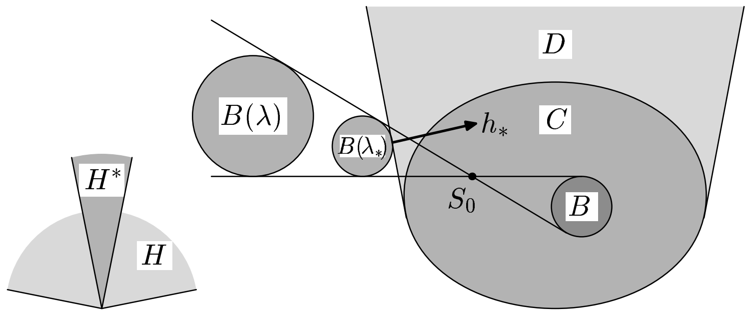
If $h \in H \setminus N$, then there are three possibilities:

1) h is orthogonal to the smallest affine subspace containing D;
2) $\sup_{x \in B(\lambda_*)} \langle h, x\rangle > \langle h, T\rangle$;
3) $\inf_{x \in D} \langle h, x\rangle < \langle h, T\rangle$.

In the first case, $\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = 0$. In the second case,

$$\inf_{x \in B} \langle h, x - S_0\rangle < \langle h, U - S_0\rangle, \qquad \inf_{x \in D} \langle h, x - S_0\rangle \leq \langle h, T - S_0\rangle,$$

so that $\mathrm{RAROC}(\langle h, S_1 - S_0\rangle) < \lambda_*^{-1}$. The third case is analyzed in a similar way. □



`Figure 3.` Solution of the optimization problem. Here h_* is an optimal h.

As a corollary, in the case, where $H = \mathbb{R}^d$ and $\mathcal{PD} = \{\mathsf{P}\}$, the solution to the optimization problem is found as follows. Let T be the intersection of the ray $(\mathsf{E}_\mathsf{P} S_1, S_0)$ with the border of C. Then

$$\sup_{h\in\mathbb{R}^d} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle) = \frac{|\mathsf{E}_\mathsf{P} S_1 - S_0|}{|S_0 - T|}$$

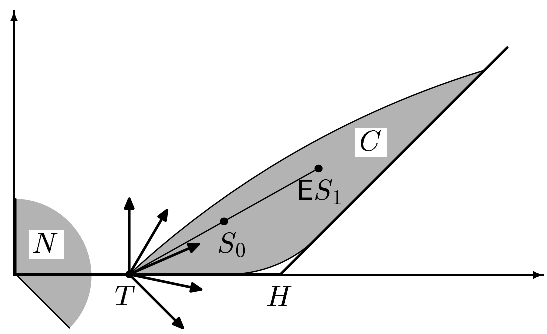
and $\operatorname{argmax}_{h\in\mathbb{R}^d} \mathrm{RAROC}(\langle h, S_1 - S_0\rangle)$ is

$$N_D(T) := \{h \in \mathbb{R}^d : \forall x \in C^\circ,\ \langle h, x - T\rangle > 0\}.$$

In the case, where C has a nonempty interior, $N_D(T)$ is the set of inner normals to C at the point T.

The following example shows that in natural situations the set of optimal strategies h_* might not be unique (of course, the uniqueness of h_* should be understood up to multiplication by a positive constant).

Example 3.2. Let S_1^1 have lognormal distribution and $S_1^2 = (S_1^1 - K)^+$ (so that the second asset is a call option on the first one). Let $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD}$ be the determining set of Tail V@R of order λ, and $H = \mathbb{R}^2$. Assume that $\mathcal{F} = \sigma(S_1^1)$. It is easy to see that $\mathcal{X}_{\mathcal{RD}}(S_1^1)$ consists of a unique element $\mathsf{Q} = \lambda^{-1} I(S_1^1 \le q_\lambda)\mathsf{P}$, where q_λ is the λ-quantile of S_1^1. The border of C has an angle $\pi/4$ at the point $E = \mathsf{E}_\mathsf{Q}(S_1^1, S_1^2)$ (see Figure 4). Let $S_0 = \frac{E + \mathsf{E}_\mathsf{P} S_1}{2}$. Then $T = E$ and $N_D(T) = \{h \in \mathbb{R}^2 : h^1 \ge 0,\ h^2 \ge -h^1\}$. □



`Figure 4.` Nonuniqueness of an optimal strategy

Let us now find the solution of the optimization problem in the Gaussian case.

Example 3.3. Let S_1 have Gaussian distribution with mean a and covariance matrix B. Let $\mathcal{PD} = \{\mathsf{P}\}$ and $\mathcal{RD}$ be the determining set of a law invariant coherent utility function u that is finite on Gaussian random variables. We consider the setting with no constraints, i.e. $H = \mathbb{R}^d$. Assume that S_0 belongs to the relative interior of C and $S_0 \neq a$.

There exists $\gamma > 0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi) = m - \gamma\sigma$. Let L denote the image of $\mathbb{R}^d$ under the map $x \mapsto Bx$. It is easy to see that

$$C = a + \{B^{1/2}x : \|x\| \le \gamma\} = a + \{y \in L : \langle y, B^{-1}y\rangle \le \gamma^2\}.$$

We have $T = a - \alpha(S_0 - a)$ with some $\alpha > 0$. It is easy to see that $h \in N_D(T)$ if and only if $\langle h, a - S_0\rangle > 0$ and, for any $y \in L$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \langle T - a + \varepsilon y, B^{-1}(T - a + \varepsilon y)\rangle = 0,$$

we have $\langle \mathrm{pr}_L h, y\rangle = 0$. This means that $\mathrm{pr}_L h = c' B^{-1}(a - T) = cB^{-1}(a - S_0)$ with some constant $c > 0$. Thus,

$$N_D(T) = \{h \in \mathbb{R}^d : Bh = c(a - S_0),\ c > 0\}.$$

Note that this set does not depend on u! □

3.3 Optimal Structure of a Firm

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD} \subseteq \mathcal{P}$ be an L^1-closed convex set, and let $X^1, \dots, X^d \in L^1_w(\mathcal{RD})$ be the discounted incomes produced by different components of some firm. Let $H^1, \dots, H^d$ be sets, each of which equals either $\mathbb{R}_+$ or $\mathbb{R}$. We will consider the problem of maximizing the RAROC over the set $A = \{\langle h, X\rangle : h \in H\}$, where $H = H^1 \times \dots \times H^d$. From the financial point of view, this means that each component of the firm is allowed to grow or shrink (the condition $H^i = \mathbb{R}$ means that the portfolio of the i-component can be reverted; the condition $H^i = \mathbb{R}_+$ means that it cannot be reverted) and the problem is to find the firm's structure that maximizes the overall RAROC. We will assume that $\mathsf{E}_\mathsf{P} X^i \neq 0$ for any i, that $0 < R_* < \infty$, where $R_* = \sup_{X \in A} \mathrm{RAROC}(X)$, and that the set C given by (3.1) is strictly convex, i.e. its interior is nonempty and its border contains no interval.

Definition 3.4. Let $\mathcal{PD}$ be a profit-determining set and $\mathcal{RD}$ be a risk-determining set. We define the *RAROC contribution of X to Y* as

$$\mathrm{RAROC}^c(X;Y) = \frac{\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_\mathsf{Q} X}{-u^c(X;Y)},$$

where u is the coherent utility function with the determining set $\mathcal{RD}$.

The RAROC contribution is well defined provided that $u^c(X;Y)$ is well defined and $u^c(X;Y) \neq 0$.

Remarks. (i) RAROC contribution may take on negative values.
(ii) We have $\mathrm{RAROC}^c(X;X) = \mathrm{RAROC}(X)$.

Theorem 3.5. *Let $h \in H$ be such that $h^i \neq 0$ for any i. Then* $\mathrm{RAROC}(\langle h, X\rangle) = R_*$ *if and only if*

$$\mathrm{RAROC}^c\Big(h^1X^1; \sum_{i=1}^d h^iX^i\Big) = \dots = \mathrm{RAROC}^c\Big(h^dX^d; \sum_{i=1}^d h^iX^i\Big). \tag{3.4}$$

Moreover, in this case all the elements of this equality are equal to R_*.

`Proof`. Let us prove the "only if" part. Introduce the notation $X = (X^1, \dots, X^d)$, $e = (1, \dots, 1)$. We will denote by xy the componentwise product of vectors x and y (i.e. $(xy)^i = x^iy^i$). The condition $R_* > 0$, combined with Theorem 3.1, shows that the ray $\{-\lambda \mathsf{E_P}X : \lambda \geq 0\}$ intersects the border of D at a point T (D is given by (3.1)). We have

$$\forall x \in D,\ \langle e, hT\rangle = \langle h, T\rangle \leq \langle h, x\rangle = \langle e, hx\rangle. \tag{3.5}$$

We can write $hT = hx_1 + hx_2$ with $x_1 \in C$, $x_2 \in H^*$ (H^* is given by (3.1)). Clearly, $hx_2 \in \mathbb{R}^d_+$. This, combined with (3.5), shows that $hx_2 = 0$. Thus,

$$hT \in hC = \{hx : x \in C\} = \{\mathsf{E_Q}hX : \mathsf{Q} \in \mathcal{RD}\}.$$

It follows from (3.5) and Theorem 1.11 that hT is a utility allocation between $h^1X^1, \dots, h^dX^d$ corresponding to the coherent utility function u with the determining set $\mathcal{RD}$. The strict convexity of hC ensures that the utility allocation between $h^1X^1, \dots, h^dX^d$ is unique (corresponding to the coherent utility function with the determining set $\mathcal{RD}$). It follows from Theorem 3.1 that $T = -R_*^{-1}\mathsf{E_P}X$. Employing now Theorem 1.11, we get

$$\mathrm{RAROC}^c\Big(h^iX^i; \sum_{i=1}^d h^iX^i\Big) = \frac{\mathsf{E_P}h^iX^i}{-u^c\big(h^iX^i; \sum_{i=1}^d h^iX^i\big)} = \frac{h^i\mathsf{E_P}X^i}{-h^iT^i} = R_*.$$

Let us prove the "if" part. Let $y_0 \in \mathbb{R}^d$ be a utility allocation between $h^1X^1, \dots, h^dX^d$ (it is unique due to the strict convexity of C and the condition $h^i \neq 0$ for any i). It follows from Theorem 1.11 that $y_0 \in hC$, so that we can write $y_0 = hT$ with some $T \in C$. For any $x_1 \in C$, $x_2 \in H^*$, we have

$$\langle h, x_1 + x_2\rangle = \langle e, hx_1 + hx_2\rangle \geq \langle e, hx_1\rangle \geq \langle e, y_0\rangle = \langle h, T\rangle$$

(in the first inequality we used the inclusion $hx_2 \in \mathbb{R}^d_+$). This means that T belongs to the border of D and $\langle h, x - T\rangle \geq 0$ for any $x \in D$. As $h \neq 0$ and D has a nonempty interior, h is an inner normal to D at the point T. It follows from (3.4) that $y_0 = \alpha\mathsf{E_P}hX$ with some $\alpha \in \mathbb{R}$, i.e. $h^iT^i = \alpha h^i\mathsf{E_P}X^i$, $i = 1, \dots, d$. As $h^i \neq 0$ for each i, we get $T = \alpha\mathsf{E_P}X$. The condition $R_* < \infty$ ensures that $0 \in D^\circ$, so that $\alpha \neq 0$. We have $\mathsf{E_P}\langle h, X\rangle = \langle h, \mathsf{E_P}X\rangle$ and $u(\langle h, X\rangle) = \langle h, T\rangle = \alpha\langle h, \mathsf{E_P}X\rangle$, and therefore, $\alpha < 0$. Now, it follows from Theorem 3.1 that $\mathrm{RAROC}(\langle h, X\rangle) = R_*$. □

`Remark`. Theorem 3.5 is not true for an arbitrary convex constraint H. As an example, one can take $H = \{h \in \mathbb{R}^d : h^1 = \dots = h^d \geq 0\}$ and $X^1, \dots, X^d$ such that the utility allocation between $X^1, \dots, X^d$ is unique and is not collinear to $\mathsf{E_P}X$. Then, for any $h \in H$, we have $\mathrm{RAROC}(\langle h, X\rangle) = R_*$, but (3.4) is violated in view of Theorem 3.5.

Consider now the case, where the set of discounted incomes that can be obtained by the i-th component of the firm is a convex cone $A^i \subseteq L^0$. Thus, the set of incomes available to the whole firm is $A = \sum_{i=1}^d A_i$. Assume that $0 < R_* < \infty$ and let $Y = \sum_{i=1}^d Y^i \in \operatorname{argmax}_{X \in A} \mathrm{RAROC}(X)$ (here $Y^i \in A^i$). Assume that $Y^i \in L^1_w(\mathcal{RD})$, $\mathsf{E_P}Y^i \neq 0$ for any i and a utility allocation between $Y^1, \dots, Y^d$ is unique.

Corollary 3.6. *We have*

$$\mathrm{RAROC}^c(Y^1;Y)=\dots=\mathrm{RAROC}^c(Y^d;Y)=R_*. \tag{3.6}$$

Proof. Note that $A\supseteq A'=\{\langle h,Y\rangle : h\in\mathbb{R}^d_+\}$, so that $\mathrm{RAROC}(Y)=\sup_{X\in A'}\mathrm{RAROC}(X)$. Now, the statement follows from Theorem 3.5. □

Remark. The reverse statement is not true. Indeed, take $X^i\in A^i$ such that $X^i\in L^1_w(\mathcal{RD})$, $\mathsf{E_P}X^i\neq 0$ for any i, and let $h\in\operatorname{argmax}_{h\in\mathbb{R}^d_+}\mathrm{RAROC}(\langle h,X\rangle)$. By Theorem 3.5, equality (3.6) is satisfied for $Y^i=h^iX^i$, but $\sum_{i=1}^d Y^i$ is not necessarily an optimal element of A because A^i can contain other elements than X^i.

3.4 Single-Agent Global Optimization

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, $A\subseteq L^0$ be a $\mathcal{D}$-consistent convex cone, and $W\in L^1_s(\mathcal{D})$. From the financial point of view, W is the current endowment of some agent, while A is the set of discounted incomes the agent can obtain by trading. We will assume that there exists no $X\in A$ with $u(X)>0$.

Problem (single-agent global optimization): Find

$$u_*=\sup_{X\in A}u(W+X)$$

and

$$X_*=\operatorname*{argmax}_{X\in A}u(W+X).$$

Theorem 3.7. *We have*

$$u_*=\inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}}\mathsf{E_Q}W.$$

Proof. By Theorem 2.4, for any $z\in\mathbb{R}$,

$$\sup_{X\in A}u(W+X)>z\iff\sup_{X\in A}u(-z+W+X)>0\iff\mathcal{D}\cap\mathcal{R}(-z+W+A)=\emptyset.$$

Fix $\mathsf{Q}\in\mathcal{R}(-z+W+A)$. As A is a cone, we have $\mathsf{E_Q}X\le 0$ for any $X\in A$. As A contains zero, $\mathsf{E_Q}(-z+W)\le 0$. Thus, $\mathsf{Q}\in\mathcal{R}$ and $\mathsf{E_Q}(-z+W)\le 0$. Conversely, if these two conditions are satisfied, then $\mathsf{Q}\in\mathcal{R}(-z+W+A)$. We get

$$\sup_{X\in A}(W+X)>z\iff\mathcal{D}\cap\mathcal{R}\cap\{\mathsf{Q}:\mathsf{E_Q}W\le z\}=\emptyset,$$

and the result follows. □

3.5 Static Model with a Finite Number of Assets

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, u be a coherent utility function with the determining set $\mathcal{D}$, and $W\in L^1_w(\mathcal{D})$. Let $A=\{\langle h,X\rangle : h\in H\}$, where $X=(X^1,\dots,X^d)\in L^1_w(\mathcal{D})$ and $H\subseteq\mathbb{R}^d$ is a closed convex cone. We will assume that there exists no $X\in A$ with $u(X)>0$.

Let us introduce the notation (see Figure 5)

$$\begin{aligned}
C &= \mathrm{cl}\{\mathsf{E}_{\mathsf{Q}}(X,W) : \mathsf{Q}\in\mathcal{D}\},\\
\widetilde{H} &= \{x\in\mathbb{R}^{d+1} : (x^1,\dots,x^d)\in H,\ x^{d+1}=1\},\\
\widetilde{H}^* &= \{x\in\mathbb{R}^{d+1} : \forall h\in\widetilde{H},\ \langle h,x\rangle\le 0\},\\
e &= (0,\dots,0,1),\\
\lambda_* &= \inf\{\lambda\in\mathbb{R} : (\lambda e+\widetilde{H}^*)\cap C\ne\emptyset\},\\
\widetilde{N} &= \{h\in\mathbb{R}^{d+1} : h^{d+1}=1 \text{ and } \exists a\in\mathbb{R} : \forall x\in\lambda_* e+\widetilde{H}^*,\ \forall y\in C,\ \langle h,x\rangle\le a\le\langle h,y\rangle\},\\
N &= \{h\in\mathbb{R}^d : (h,1)\in\widetilde{N}\}.
\end{aligned}$$

Note that $C\cap\{\alpha e : \alpha\in\mathbb{R}\}\ne\emptyset$. Indeed, otherwise there exists $\widetilde{h}=(h,1)\in\mathbb{R}^{d+1}$ such that $\inf_{x\in C}\langle\widetilde{h},x\rangle>0$, which means that $u(W+\langle h,X\rangle)>0$. Furthermore, $\widetilde{H}^*\supseteq\{\alpha e:\alpha\le 0\}$ and $\widetilde{H}^*\cap\{\alpha e:\alpha>0\}=\emptyset$. Hence, $\lambda_*\in(-\infty,\infty)$.

Theorem 3.8. *We have* $u_*=\lambda_*$ *and* $\operatorname{argmax}_{h\in H} u(W+\langle h,X\rangle)=N$.

Proof. Fix $\lambda<\lambda_*$. As C is a convex compact and $\widetilde{H}^*$ is convex and closed, there exist $\widetilde{h}\in\mathbb{R}^{d+1}$ and $a,b\in\mathbb{R}$ such that, for any $x\in\lambda e+\widetilde{H}^*$ and any $y\in C$, we have $\langle\widetilde{h},x\rangle\le a<b\le\langle\widetilde{h},y\rangle$. As C is compact, $\widetilde{h}$ can be chosen in such a way that $\widetilde{h}^{d+1}\ne 0$. Since $\widetilde{H}^*\supseteq\{\alpha e:\alpha\le 0\}$, we have $\widetilde{h}^{d+1}>0$. Without loss of generality, $\widetilde{h}^{d+1}=1$. Then, for any $x\in\widetilde{H}^*$, we have $\langle\widetilde{h},x\rangle\le a-\lambda$. As $\widetilde{H}^*$ is a cone, for any $x\in\widetilde{H}^*$, we have $\langle\widetilde{h},x\rangle\le 0$ and $a-\lambda\ge 0$. As $\widetilde{h}^{d+1}=1$, we have $\widetilde{h}\in\widetilde{H}$. Let h be a d-dimensional vector that consists of the first d components of $\widetilde{h}$. Using the closedness of H, one can check that $h\in H$. Furthermore,

$$u(W+\langle h,X\rangle)=\inf_{\mathsf{Q}\in\mathcal{D}}\mathsf{E}_{\mathsf{Q}}(W+\langle h,X\rangle)=\inf_{x\in C}\langle\widetilde{h},x\rangle>\lambda.$$

As $\lambda<\lambda_*$ has been chosen arbitrarily, we conclude that $\sup_{h\in H}u(W+\langle h,X\rangle)\ge\lambda_*$.

Let $x_0\in(\lambda_* e+\widetilde{H}^*)\cap C$. Fix $h\in H$ and set $\widetilde{h}=(h,1)$. Then

$$u(W+\langle h,X\rangle)=\inf_{x\in C}\langle\widetilde{h},x\rangle\le\langle\widetilde{h},x_0\rangle.$$

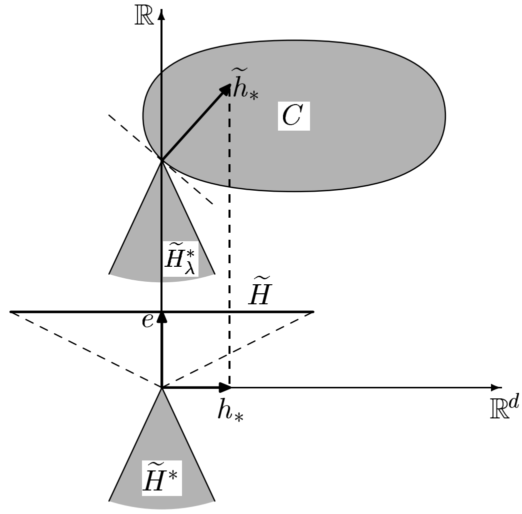
We can write $x_0=\lambda_* e+z_0$ with $z_0\in\widetilde{H}^*$. Then $\langle\widetilde{h},x_0\rangle=\lambda_*+\langle\widetilde{h},z_0\rangle\le\lambda_*$. Thus, $\sup_{h\in H}u(W+\langle h,X\rangle)\le\lambda_*$. As a result, $u_*=\lambda_*$.

Let $h\in N$. Using the same arguments as above, we show that $h\in H$. For $\widetilde{h}=(h,1)$, there exists $a\in\mathbb{R}$ such that, for any $x\in\lambda_* e+\widetilde{H}^*$ and any $y\in C$, we have $\langle h,x\rangle\le a\le\langle h,y\rangle$. The same arguments as above show that $a\ge\lambda_*$. Consequently,

$$u(W+\langle h,X\rangle)=\inf_{x\in C}\langle\widetilde{h},x\rangle\ge a\ge\lambda_*.$$

Let $h\in H$ be such that $u(W+\langle h,X\rangle)=\lambda_*$. This means that, for $\widetilde{h}=(h,1)$, we have $\inf_{x\in C}\langle\widetilde{h},x\rangle\ge\lambda_*$. Furthermore, for any $x=\lambda_* e+z\in\lambda_* e+\widetilde{H}^*$, we have $\langle\widetilde{h},x\rangle=\langle\widetilde{h},\lambda_* e\rangle+\langle\widetilde{h},z\rangle\le\lambda_*$. Thus, $\widetilde{h}\in\widetilde{N}$, which means that $h\in N$. □

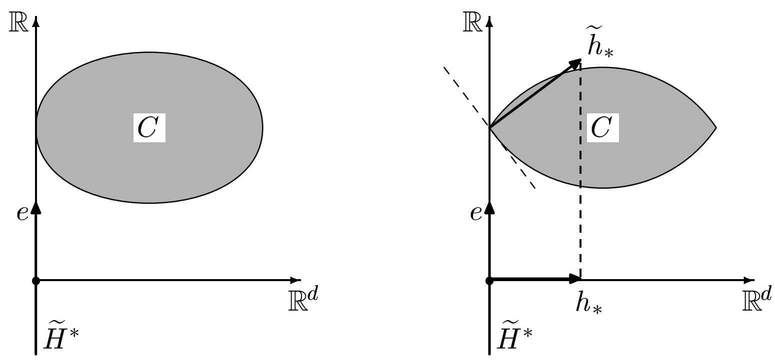
Example 3.9. (i) Let $H=\mathbb{R}^d$. Then $\widetilde{H}=\{e\}$, $\widetilde{H}^*=\{\alpha e:\alpha\le 0\}$, and $\lambda_*=\inf\{x^{d+1}:x\in C_0\}$, where $C_0=C\cap(\{0\}\times\mathbb{R})$. The condition that there exists no $X\in A$ with $u(X)>0$ is equivalent to: $C_0\ne\emptyset$. If $C^\circ\cap(\{0\}\times\mathbb{R})\ne\emptyset$, where



`Figure 5.` Solution of the optimization problem. By $\widetilde{H}^*_\lambda$ we denote $\lambda_* e + \widetilde{H}^*$. Here $\widetilde{N} = \{\widetilde{h}_*\}$ and $N = \{h_*\}$.

C° denotes the relative interior of C, then $N \neq \emptyset$ (see Figure 5). If $C^\circ \cap (\{0\} \times \mathbb{R}) = \emptyset$, then both cases $N \neq \emptyset$ and $N = \emptyset$ are possible (see Figure 6).

(ii) Let $H = \mathbb{R}^d_+$. Then $\widetilde{H} = \mathbb{R}^d_+ \times \{1\}$, $\widetilde{H}^* = (-\infty, 0]^{d+1}$, and $\lambda_* = \inf\{x^{d+1} : x \in C_-\}$, where $C_- = C \cap ((-\infty, 0]^d \times \mathbb{R})$. □



`Figure 6.` Existence (right) and nonexistence (left) of an optimal strategy for the case $H = \mathbb{R}^d$

3.6 Single-Agent Local Optimization

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, $A \subset L^0$ be a $\mathcal{D}$-consistent convex set containing zero, and $W \in L^1_s(\mathcal{D})$. The financial interpretation is the same as above. As opposed to Subsection 3.4, we assume that $\sup_{X \in A,\, \mathsf{Q} \in \mathcal{D}} |\mathsf{E}_\mathsf{Q} X| < \infty$ and $A \subseteq L^1_s(\mathcal{D})$.

Problem (single-agent local optimization): Find

$$u_* = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left[\sup_{X \in A} u(W + \varepsilon X) - u(W) \right]$$

and an element $X_* \in A$, for which

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1}[u(W+\varepsilon X_*) - u(W)] = u_*. \tag{3.7}$$

The theorem below shows that the problem posed above is equivalent to the problem of maximizing $u^c(X;W)$ over A, where $u^c(\,\cdot\,;W)$ is defined on L^0 by (1.11).

Theorem 3.10. *We have* $u_* = \sup_{X\in A} u^c(X;W)$. *Furthermore,* X_* *solves* (3.7) *if and only if* $X_* \in \operatorname{argmax}_{X\in A} u^c(X;W)$.

Proof. Theorem 1.15, combined with the inequality

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}\Big(\sup_{X\in A} u(W+\varepsilon X) - u(W)\Big)\\
&\le \limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}\Big(\sup_{X\in A}\Big(\inf_{\mathsf{Q}\in\mathcal{X}_{\mathcal{D}}(W)} \mathsf{E}_{\mathsf{Q}}(W+\varepsilon X) - u(X)\Big)\Big)\\
&= \sup_{X\in A} u^c(X;W),
\end{aligned}$$

shows that $u_* = \sup_{X\in A} u^c(X;W)$.

The second statement follows immediately from Theorem 1.15. □

3.7 Static Model with a Finite Number of Assets

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space and u be a coherent utility function with the determining set $\mathcal{D}$. Let $A = \{\langle h, X\rangle : h \in H\}$, where $X = (X^1,\dots,X^d) \in L^1_w(\mathcal{D})$ and $H \subset \mathbb{R}^d$ is a convex compact. We will consider the problem of maximizing $u(X)$ over A.

Let us introduce the notation (see Figure 7)

$$\begin{aligned}
C &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} X : \mathsf{Q}\in\mathcal{D}\},\\
H^* &= \{x\in\mathbb{R}^d : \forall h\in H,\ \langle h,x\rangle \le 1\},\\
\lambda_* &= \inf\{\lambda\ge 0 : \lambda H^*\cap C \ne \emptyset\},\\
N &= \begin{cases} \{h : \forall x\in\lambda_* H^*,\ \forall y\in C,\ \langle h,x\rangle \le \lambda_* \le \langle h,y\rangle\} & \text{if } \lambda_*>0,\\ \emptyset & \text{if } \lambda_*=0.\end{cases}
\end{aligned}$$

Note that $\lambda_* < \infty$, $N \ne \emptyset$, and $N \subseteq H$.

Theorem 3.11. *We have* $\sup_{X\in A} u(X) = \lambda_*$ *and* $\operatorname{argmax}_{h\in H} u(\langle h, X\rangle) = N$.

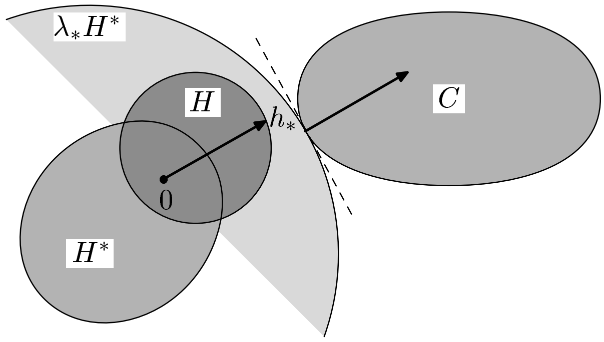
Proof. Let $\lambda_* > 0$. For $h \in N$, we have

$$u(\langle h, X\rangle) = \inf_{x\in C}\langle h,x\rangle = \lambda_*.$$

For $h \in H\setminus N$, we have $\sup_{x\in\lambda_* H^*}\langle h,x\rangle \le \lambda_*$, and consequently,

$$u(\langle h, x\rangle) = \inf_{x\in C}\langle h,x\rangle < \lambda_*.$$

The case $\lambda_* = 0$ is analyzed trivially. □



`Figure 7.` Solution of the optimization problem. Here h_* is the optimal h.

3.8 Liquidity Effects in NGD Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, and $A \subset L^0$ be a convex set containing zero. We assume that there exists no $X \in A$ with $u(X) > 0$.

Definition 3.12. We define the *upper* and *lower utility-based NGD price functions* of a contingent claim F as

$$\overline{P}_F(v) = \sup\{x : \text{the model } (\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A - v(F - x)) \text{ satisfies the NGD}\}, \quad v > 0,$$
$$\underline{P}_F(v) = \inf\{x : \text{the model } (\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}, A + v(F - x)) \text{ satisfies the NGD}\}, \quad v > 0.$$

From the financial point of view, v means the value of a trade.

In view of the equality $\underline{P}_F(v) = -\overline{P}_{-F}(v)$, it is sufficient to study only the properties of $\overline{P}_F$.

Theorem 3.13. *Assume that* $F \in L^1_s(\mathcal{D})$.
(i) *The function* $\overline{P}_F$ *is increasing and continuous.*
(ii) *We have*

$$\lim_{v \downarrow 0} \overline{P}_F(v) = \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_\mathsf{Q} F.$$

(iii) *We have*

$$\lim_{v \to \infty} \overline{P}_F(v) \le \sup_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} F.$$

If $\sup_{X \in A,\, \mathsf{Q} \in \mathcal{D}} |\mathsf{E}_\mathsf{Q} X| < \infty$, *then*

$$\lim_{v \to \infty} \overline{P}_F(v) = \sup_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_\mathsf{Q} F.$$

`Proof.` **(i)** It follows from the equality

$$\sup_{X \in A} u(-v(F - x) + X) = vx + \sup_{X \in A} u(-vF + X)$$

that $\overline{P}_F(v) = -v^{-1} f(v)$, where $f(v) = \sup_{X \in A} u(-vF + X)$. Note that f is finite due to the NGD and the condition $F \in L^1_s(\mathcal{D})$. Fix $v_1, v_2 > 0$, $\varepsilon > 0$, $\alpha \in [0, 1]$ and find

$X_1, X_2 \in A$ such that $u(-v_iF + X_i) \ge f(v_i) - \varepsilon$, $i = 1, 2$. Then

$$\begin{aligned} f(\alpha v_1 + (1-\alpha)v_2) &\ge u(-(\alpha v_1 + (1-\alpha)v_2)F + \alpha X_1 + (1-\alpha)X_2) \\ &\ge \alpha u(-v_1F + X_1) + (1-\alpha)u(-v_2F + X_2) \\ &\ge \alpha f(v_1) + (1-\alpha)f(v_2) - \varepsilon. \end{aligned}$$

Consequently, f is concave. As A contains zero and the NGD is satisfied, we have $f(0) = 0$. This leads to the desired statement.

(ii) By Theorem 3.7,

$$\sup_{X \in \text{cone } A} u(-vF + X) = \inf_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}}(-vF) = -v \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F,$$

where "cone" denotes the cone hull. Take $\varepsilon > 0$ and find $X_0 \in A$, $\alpha_0 \ge 0$ such that

$$u(-F + \alpha_0 X_0) \ge -\sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F - \varepsilon.$$

As the function $\mathbb{R}_+ \ni x \mapsto u(-xF + x\alpha_0 X_0)$ is concave and vanishes at zero, we have

$$u(-vF + v\alpha_0 X_0) \ge v\biggl(-\sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F - \varepsilon\biggr), \quad v \le 1.$$

As $\varepsilon > 0$ has been chosen arbitrarily, we get

$$\limsup_{v \downarrow 0} \overline{P}_F(v) = \limsup_{v \downarrow 0}(-v^{-1} f(v)) \le \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F.$$

Combining this with the inequality

$$\sup_{X \in A} u(-vF + X) \le \sup_{X \in A} \inf_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}}(-vF + X) = \inf_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}}(-vF) = -v \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} F,$$

we get the desired statement.

(iii) The first statement follows from the inequality

$$\sup_{X \in A} u(-vF + X) \ge u(-vF) = -v \sup_{\mathsf{Q} \in \mathcal{D}} \mathsf{E}_{\mathsf{Q}} F.$$

The second statement is an obvious consequence of the equality $\overline{P}_F(v) = -\sup_{X \in A} u(-F + v^{-1}X)$. □

`Remarks`. `(i)` If A is a cone, then, clearly, $\overline{P}_F = \mathit{const}$.

`(ii)` If $\sup_{X \in A, \mathsf{Q} \in \mathcal{D}} |\mathsf{E}_{\mathsf{Q}} X| < \infty$, then

$$\overline{P}_F(\infty) - \underline{P}_F(\infty) = \sup_{\mathsf{Q} \in \mathcal{D}} F - \inf_{\mathsf{Q} \in \mathcal{D}} F,$$

which is the length of the NGD price interval in the absence of a market. The difference

$$\overline{P}_F(0) - \underline{P}_F(0) = \sup_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} F - \inf_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} F$$

is the length of the NGD price interval in the presence of a market. Thus, the ratio

$$\frac{\overline{P}_F(0) - \underline{P}_F(0)}{\overline{P}_F(\infty) - \underline{P}_F(\infty)}$$

measures the "closeness" of a new instrument F to those already existing in the market.

Example 3.14. Consider a static model with a finite number of assets, i.e. $A = \{\langle h, X\rangle : h \in H\}$, where $X = (X^1, \dots, X^d) \in L^1_w(\mathcal{D})$ and $H \subset \mathbb{R}^d$ is a convex bounded set. Assume that H contains a neighborhood of zero. Denote $C = \{\mathsf{E}_\mathsf{Q}(X, F) : \mathsf{Q} \in \mathcal{D}\}$. Then

$$\begin{aligned}\overline{P}_F(0) &= \sup\{x^{d+1} : x^1 = \dots = x^d = 0,\ x \in C\},\\ \overline{P}_F(\infty) &= \sup\{x^{d+1} : x \in C\}.\end{aligned}$$

Note that these values do not depend on H! □

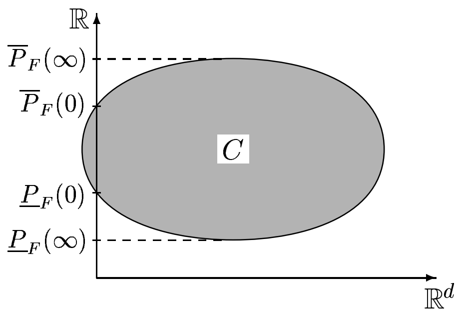


Figure 8. The form of $\overline{P}_F(0)$, $\overline{P}_F(\infty)$, $\underline{P}_F(0)$, and $\underline{P}_F(\infty)$

4 Optimality Pricing

4.1 Agent-Independent Optimality Pricing

Consider the model of Subsection 2.2. Assume that $0 < R_* < \infty$, where $R_* = \sup_{X \in A} \mathrm{RAROC}(X)$. It follows from Theorem 2.10 that

$$R_* = \inf\Big\{R > 0 : \Big(\frac{1}{1+R}\mathcal{PD} + \frac{R}{1+R}\mathcal{RD}\Big) \cap \mathcal{R} \neq \emptyset\Big\}$$

and $\mathcal{D}_* \cap \mathcal{R} \neq \emptyset$, where $\mathcal{D}_* = \frac{1}{1+R_*}\mathcal{PD} + \frac{R_*}{1+R_*}\mathcal{RD}$.

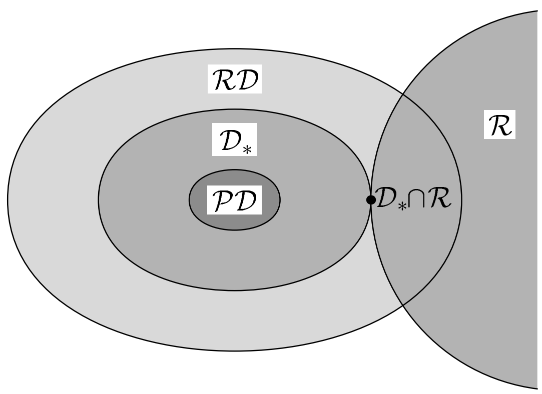


Figure 9. The structure of $\mathcal{D}_* \cap \mathcal{R}$

Definition 4.1. An *agent-independent NBC price* of a contingent claim F is a real number x such that

$$\sup_{X\in A+A(x)} \mathrm{RAROC}(X) = \sup_{X\in A} \mathrm{RAROC}(X),$$

where $A(x) = \{h(F-x) : h \in \mathbb{R}\}$.

The set of NBC prices will be denoted by $I_{NBC}(F)$.

Theorem 4.2. *For $F \in L^1_s(\mathcal{RD})$,*

$$I_{NBC}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}\}.$$

`Proof`. If $x \in I_{NBC}(F)$, then, by Theorem 2.10, there exists $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}(A + A(x))$. This means that $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}$ and $\mathsf{E}_\mathsf{Q} F = x$.

Conversely, if $x = \mathsf{E}_\mathsf{Q} F$ with some $\mathsf{Q} \in \mathcal{D}_* \cap \mathcal{R}$, then, for any $X + h(F-x) \in A + A(x)$, we have $\mathsf{E}_\mathsf{Q} X \le 0$, so that $\mathsf{Q} \in \mathcal{R}(A + A(x))$. Due to Theorem 2.10, $\sup_{X\in A+A(x)} \mathrm{RAROC}(X) \le R_*$. □

The following lemma yields a more definite representation of $\mathcal{D}_* \cap \mathcal{R}$.

Lemma 4.3. *If $X_* \in \operatorname{argmax}_{X\in A} \mathrm{RAROC}(X)$, then*

$$\mathcal{D}_* \cap \mathcal{R} = \left(\frac{1}{1+R_*}\,\mathcal{X}_{\mathcal{PD}}(X_*) + \frac{R_*}{1+R_*}\,\mathcal{X}_{\mathcal{RD}}(X_*)\right) \cap \mathcal{R}. \tag{4.1}$$

`Proof`. Take

$$\mathsf{Q} = \frac{1}{1+R_*}\,\mathsf{Q}_1 + \frac{R_*}{1+R_*}\,\mathsf{Q}_2 \in \mathcal{D}_* \cap \mathcal{R}.$$

We have

$$\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_\mathsf{Q} X_* + R_* \inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E}_\mathsf{Q} X_* \le \mathsf{E}_{\mathsf{Q}_1} X_* + R_* \mathsf{E}_{\mathsf{Q}_2} X_* \le 0$$

(the second inequality follows from the inclusion $\mathsf{Q} \in \mathcal{R}$). Combining this with the equality

$$\mathrm{RAROC}(X_*) = \frac{\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_\mathsf{Q} X_*}{-\inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E}_\mathsf{Q} X_*} = R_*,$$

we get

$$\inf_{\mathsf{Q}\in\mathcal{PD}} \mathsf{E}_\mathsf{Q} X_* + R_* \inf_{\mathsf{Q}\in\mathcal{RD}} \mathsf{E}_\mathsf{Q} X_* \le \mathsf{E}_{\mathsf{Q}_1} X_* + R_* \mathsf{E}_{\mathsf{Q}_2} X_*.$$

This means that $\mathsf{Q}_1 \in \mathcal{X}_{\mathcal{PD}}(X_*)$ and $\mathsf{Q}_2 \in \mathcal{X}_{\mathcal{RD}}(X_*)$. □

As a corollary, if $\mathcal{X}_{\mathcal{PD}}(X_*)$ and $\mathcal{X}_{\mathcal{RD}}(X_*)$ are singletons (this is true, for instance, if $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD}$ is the determining set of Weighted V@R, and X_* has a continuous distribution), then $\mathcal{R}$ can be removed from (4.1), i.e.

$$\mathcal{D}_* \cap \mathcal{R} = \frac{1}{1+R_*}\,\mathcal{X}_{\mathcal{PD}}(X_*) + \frac{R_*}{1+R_*}\,\mathcal{X}_{\mathcal{RD}}(X_*).$$

But in general this equality might be violated as shown by the example below.

Example 4.4. Let $\mathcal{PD} = \{\mathsf{P}\}$, $\mathcal{RD}$ be the determining set of Tail V@R with $\lambda < 1/2$, and X^1, X^2 be independent random variables with $\mathsf{P}(X^1 = -1) = \mathsf{P}(X^1 = 2) = 1/2$, $\mathsf{P}(X^2 = \pm 1) = 1/2$. Let $A = \{h^1 X^1 + h^2 X^2 : h^i \in \mathbb{R}\}$. For any (h^1, h^2) with $h^1 \geq 0$, we have

$$\inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_{\mathsf{Q}}(h^1 X^1 + h^2 X^2) \leq \mathsf{E}_{\mathsf{P}} Z(h^1 X^1 + h^2 X^2) = h^1 \mathsf{E}_{\mathsf{P}} Z X^1 = \inf_{\mathsf{Q} \in \mathcal{RD}} \mathsf{E}_{\mathsf{Q}} X^1,$$

where $Z = 2I(X = -1)$. Combining this with the equality $\mathsf{E}_{\mathsf{P}}(h^1 X^1 + h^2 X^2) = h^1 \mathsf{E}_{\mathsf{P}} X^1$, we get that $X^1 \in \operatorname{argmax}_{X \in A} \operatorname{RAROC}(X)$. On the other hand, there exists $\mathsf{Q} \in \mathcal{X}_{\mathcal{RD}}(X^1)$, for which $\mathsf{E}_{\mathsf{Q}} X^2 \neq 0$. Thus, the set $\frac{1}{1+R_*} \mathcal{X}_{\mathcal{PD}} + \frac{R_*}{1+R_*} \mathcal{X}_{\mathcal{RD}}$ contains measures that do not belong to $\mathcal{R}$. □

4.2 Static Model with a Finite Number of Assets

Consider the model of Subsection 3.2. Assume that $0 < R_* < \infty$. Let $F \in L^1_w(\mathcal{D})$ be a contingent claim.

Let us introduce the notation (see Figure 10)

$$\begin{aligned}
H^* &= \{x \in \mathbb{R}^d : \forall h \in H,\ \langle h, x \rangle \geq 0\}, \\
\widetilde{H}^* &= H^* \times \{0\}, \\
\widetilde{B} &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}}(S_1, F) : \mathsf{Q} \in \mathcal{PD}\}, \\
\widetilde{C} &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}}(S_1, F) : \mathsf{Q} \in \mathcal{RD}\}, \\
\widetilde{D} &= \widetilde{C} + \widetilde{H}^*, \\
\widetilde{D}_R &= \frac{1}{1+R}\widetilde{B} + \frac{R}{1+R}\widetilde{D}.
\end{aligned}$$

Theorem 4.5. *We have*

$$R_* = \inf\{R > 0 : \widetilde{D}_R \cap (\{S_0\} \times \mathbb{R}) \neq \emptyset\}, \tag{4.2}$$

$$I_{NBC}(F) = \{x : (S_0, x) \in \widetilde{D}_{R_*}\}. \tag{4.3}$$

Proof. Denote

$$\begin{aligned}
B &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{PD}\}, \\
C &= \operatorname{cl}\{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{RD}\}, \\
D &= C + H^*, \\
D_R &= \frac{1}{1+R} B + \frac{R}{1+R} D.
\end{aligned}$$

Note that $B = \operatorname{pr}_{\mathbb{R}^d} \widetilde{B}$, $C = \operatorname{pr}_{\mathbb{R}^d} \widetilde{C}$, $H^* = \operatorname{pr}_{\mathbb{R}^d} \widetilde{H}^*$, and consequently, $D = \operatorname{pr}_{\mathbb{R}^d} \widetilde{D}$, $D_R = \operatorname{pr}_{\mathbb{R}^d} \widetilde{D}_R$. Combining this with the results of Subsection 3.2, we get

$$R_* = \inf\{R > 0 : D_R \ni S_0\} = \inf\{R > 0 : \widetilde{D}_R \cap (\{S_0\} \times \mathbb{R}) \neq \emptyset\}.$$

Furthermore, for any $x \in \mathbb{R}$,

$$\sup_{A + A(x)} \operatorname{RAROC}(X) = \inf\{R > 0 : \widetilde{D}_R \ni (S_0, x)\}.$$

This, combined with (4.2), proves (4.3). □

To conclude this subsection, we find the form of $I_{NBC}(F)$ in the Gaussian case.

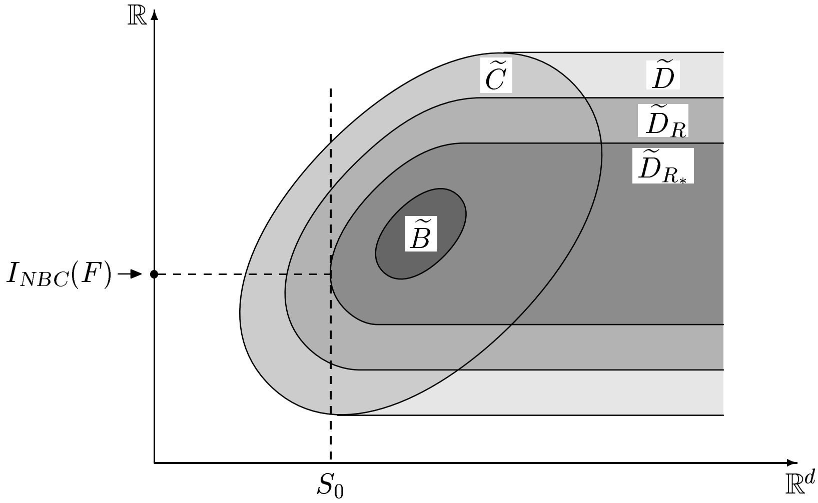


Figure 10. The form of I_{NBC}.
Here $I_{NBC}(F)$ consists of one point.

Example 4.6. Consider the setting of Example 2.13. Clearly, R_* is the solution of the equation $\langle S_0-a, B^{-1}(S_0-a)\rangle = \frac{\gamma^2 R_*^2}{1+R_*^2}$. This, combined with (2.3), shows that $I_{NBC}(F)$ consists of a unique point $\langle b, S_0-a\rangle + f$. Let us remark that this value coincides with the fair price of F obtained as a result of the mean-variance hedging. Note that this value does not depend on u! □

4.3 Single-Agent Optimality Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, u be a coherent utility function with the weakly compact determining set $\mathcal{D}$, $A \subseteq L^0$ be a $\mathcal{D}$-consistent convex set containing zero, and $W \in L^1_s(\mathcal{D})$. The financial interpretation is the same as in Subsection 3.4.

Definition 4.7. A *single-agent NBC price* of a contingent claim F is a real number x such that

$$\max_{X\in A,\, h\in\mathbb{R}} u(W+X+h(F-x)) = u(W).$$

The set of NBC prices will be denoted by $I_{NBC}(F)$.

Theorem 4.8. *For* $F \in L^1_s(\mathcal{D})$,

$$I_{NBC}(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{X}_\mathcal{D}(W) \cap \mathcal{R}\}. \tag{4.4}$$

Remark. The set of NBC prices is nonempty only if W is optimal in the sense that $\max_{X\in A} u(W+X) = u(W)$. However, if W is not optimal, then, as seen from the proof of Theorem 4.8, $\mathcal{X}_\mathcal{D}(W) \cap \mathcal{R} = \emptyset$, so that (4.4) still holds true.

Proof of Theorem 4.8. As A contains zero and the function $\mathbb{R}_+ \ni \alpha \mapsto u(W+\alpha X)$ is concave for a fixed X, the condition $x \in I_{NBC}(F)$ is equivalent to:

$$\max_{X\in \operatorname{cone} A,\, h\in\mathbb{R}} u(W+X+h(F-x)) = u(W).$$

By Theorem 3.7, this is equivalent to:

$$\inf_{\mathsf{Q}\in\mathcal{D}\cap\mathcal{R}(A+A(x))} \mathsf{E}_\mathsf{Q} W = \inf_{\mathsf{Q}\in\mathcal{D}} \mathsf{E}_\mathsf{Q} W,$$

where $A(x) = \{h(F - x) : h \in \mathbb{R}\}$. Clearly, the latter condition is equivalent to: $\mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R}(A + A(x)) \neq \emptyset$. It is easy to verify that this is equivalent to: $x = \mathsf{E}_{\mathsf{Q}} F$ for some $\mathsf{Q} \in \mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R}$. □

Let us now provide a geometric representation of $I_{NBC}(F)$ (see Figure 11). Assume that $u(W) = \max_{X \in A} u(W + X)$ (the reasoning used above shows that this is equivalent to: $\mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R} \neq \emptyset$). Consider the set $C = \{\mathsf{E}_{\mathsf{Q}}(F, W) : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$ and the function $f(x) = \inf\{y : (x, y) \in C\}$ (we set $\inf \emptyset = +\infty$).

Corollary 4.9. *For $F \in L^1_s(\mathcal{D})$,*

$$I_{NBC}(F) = \operatorname*{argmin}_{x \in \mathbb{R}} f(x).$$

Proof. It is sufficient to note that

$$\min_{x \in \mathbb{R}} f(x) = \min_{\mathsf{Q} \in \mathcal{D} \cap \mathcal{R}} \mathsf{E}_{\mathsf{Q}} W = u(W)$$

and

$$f(x) = \inf\{\mathsf{E}_{\mathsf{Q}} W : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}(A + A(x))\},$$

where $A(x) = \{h(F - x) : h \in \mathbb{R}\}$. Thus, $x \in \operatorname{argmin}_{x \in \mathbb{R}} f(x)$ if and only if $\mathcal{X}_{\mathcal{D}}(W) \cap \mathcal{R}(A + A(x)) \neq \emptyset$, which, in view of Theorem 4.8, is equivalent to the inclusion $x \in I_{NBC}(F)$. □

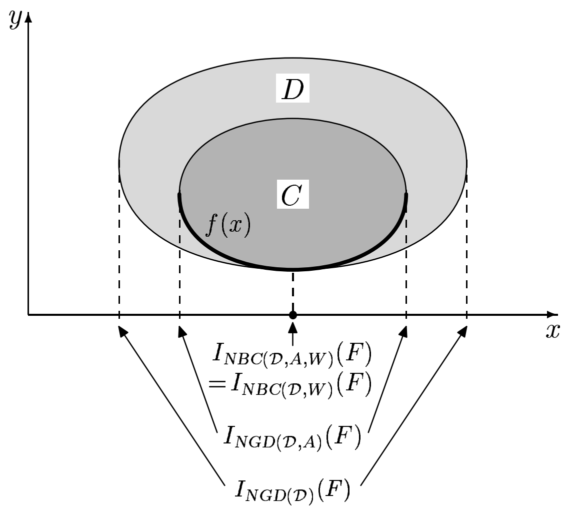


Figure 11. Comparison of various price intervals. Here $C = \{\mathsf{E}_{\mathsf{Q}}(F, W) : \mathsf{Q} \in \mathcal{D} \cap \mathcal{R}\}$ and $D = \{\mathsf{E}_{\mathsf{Q}}(F, W) : \mathsf{Q} \in \mathcal{D}\}$. In this example, $I_{NBC(\mathcal{D},A,W)}(F) = I_{NBC(\mathcal{D},W)}(F)$.

Assume that W is optimal in the sense that

$$u(W) = \max_{X \in A} u(W + X) \tag{4.5}$$

and suppose moreover that the set $I_{NBC(\mathcal{D},W)}(F)$ of NBC prices based on $\mathcal{D}$ and W (with $A = 0$) consists of one point x_0 (this condition is satisfied if the set $D = \{\mathsf{E}_{\mathsf{Q}}(F, W) : \mathsf{Q} \in \mathcal{D}\}$ is strictly convex; see Figure 11). It is seen from the proof of Theorem 4.8 that condition (4.5) is equivalent to: $\mathcal{X}_{\mathcal{D}}(W)\cap\mathcal{R}\neq\emptyset$. Then it follows from Theorem 4.8 that $I_{NBC(\mathcal{D},A,W)}(F)\neq\emptyset$ (we assume that $F\in L^1_s(\mathcal{D})$). Clearly, $I_{NBC(\mathcal{D},A,W)}(F)\subseteq I_{NBC(\mathcal{D},W)}(F)$. As a result, $I_{NBC(\mathcal{D},A,W)}(F)=\{x_0\}$. So, in this situation A can be eliminated. This situation occurs naturally as shown, in particular, by the example below.

Example 4.10. Let u be a law invariant coherent utility function that is finite on Gaussian random variables. Assume that $u(W)=\max_{X\in A}u(W+X)$ and that (W,F) has a Gaussian distribution.

There exists $\gamma>0$ such that, for a Gaussian random variable ξ with mean m and variance σ^2, we have $u(\xi)=m-\gamma\sigma$. Clearly, $I_{NBC}(F)\subseteq J$, where J is the NBC price based on $\mathcal{D}$ and W with $A=0$. Using Corollary 4.9, we deduce that J consists of a single point $\mathsf{E}F-\gamma\frac{\mathsf{cov}(F,W)}{(\mathsf{D}W)^{1/2}}$. As $I_{NBC}(F)$ is nonempty, it consists of the same point. □

4.4 Multi-Agent Optimality Pricing

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, $u_1,\dots,u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1,\dots,\mathcal{D}_N$, $A\subseteq L^0$ be a convex set containing zero, and $W_1\in L^1_s(\mathcal{D}_1),\dots,W_N\in L^1_s(\mathcal{D}_N)$. From the financial point of view, u_n, A, and W_n are the coherent utility function, the set of attainable incomes, and the current endowment of the n-th agent, respectively. We will assume that there exists a set $A'\subseteq\bigcap_n L^1_s(\mathcal{D}_n)\cap A$ such that, for any n, $\mathcal{D}_n\cap\mathcal{R}=\mathcal{D}_n\cap\mathcal{R}(A')$. We also assume that each W_n is optimal in the sense that $u_n(W_n)=\max_{X\in A}u_n(W_n+X)$.

Definition 4.11. A real number x is a *multi-agent NBC price* of a contingent claim F if there exists no element $X\in A+\{h(F-x):h\in\mathbb{R}\}$ such that $u_n(W_n+X)>u(W_n)$ for any n.

The set of NBC prices will be denoted by $I_{NBC}(F)$.

Theorem 4.12. *For* $F\in\bigcap_n L^1_s(\mathcal{D}_n)$,

$$I_{NBC}(F)=\operatorname{conv}_{n=1}^N I_{NBC(\mathcal{D}_n,A,W_n)}(F)=\bigl\{\mathsf{E}_\mathsf{Q}F:\mathsf{Q}\in\operatorname{conv}_{n=1}^N(\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R})\bigr\},$$

where $I_{NBC(\mathcal{D}_n,A,W_n)}(F)$ *is the interval of single-agent NBC prices based on* $\mathcal{D}_n$, A, W_n.

`Proof`. Let $x\in I_{NBC}(F)$. Fix $X_1,\dots,X_M\in A'$. It follows from the weak continuity of the maps $\mathcal{D}_n\ni\mathsf{Q}\mapsto\mathsf{E}_\mathsf{Q}(X_1,\dots,X_M,F)$ that, for each $n=1,\dots,N$, the set $C_n=\{\mathsf{E}_\mathsf{Q}(X_1,\dots,X_M,F-x):\mathsf{Q}\in\mathcal{X}_n\}$, where $\mathcal{X}_n=\mathcal{X}_{\mathcal{D}_n}(W_n)$, is compact. Clearly, C_n is convex. Suppose that

$$\bigl(\operatorname{conv}_{n=1}^N C_n\bigr)\cap((-\infty,0]^M\times\{0\})=\emptyset.$$

Then there exists $h\in\mathbb{R}^{M+1}$ such that $h_1,\dots,h_M\geq 0$ and $\inf_{x\in C_n}\langle h,x\rangle>0$ for each n. This means that $\inf_{\mathsf{Q}\in\mathcal{X}_n}\mathsf{E}_\mathsf{Q}Y>0$ for each n, where $Y=h^1X_1+\dots+h^mX_m+h^{m+1}(F-x)$. Employing Theorem 1.15, we conclude that there exists $\varepsilon>0$ such that $u(W_n+\varepsilon Y)>u(W_n)$ for any n.

The obtained contradiction shows that, for any $X_1,\dots,X_M\in A'$, the set

$$B(X_1,\dots,X_M)=\Biggl\{\alpha_1,\dots,\alpha_N,\mathsf{Q}_1,\dots,\mathsf{Q}_N\in S\times\prod_{n=1}^N\mathcal{X}_n:\sum_{n=1}^N\alpha_n\mathsf{E}_{\mathsf{Q}_n}F=x$$
$$\text{and }\forall n=1,\dots,N,\ \forall m=1,\dots,M,\ \mathsf{E}_{\mathsf{Q}_n}X_m\leq 0\Biggr\},$$

where $S = \{\alpha_1, \dots, \alpha_N \geq 0 : \sum_{n=1}^N \alpha_n = 1\}$, is nonempty. As the map $\mathcal{X}_n \ni \mathsf{Q} \mapsto \mathsf{E}_\mathsf{Q} X$ is weakly continuous for each $X \in L^1_s(\mathcal{D}_n)$, the set $B(X_1, \dots, X_M)$ is closed with respect to the product of weak topologies. Furthermore, any finite intersection of sets of this form is nonempty. Tikhonov's theorem, ensures that $S \times \prod_n \mathcal{X}_n$ is compact. Consequently, there exists a collection $\alpha_1, \dots, \alpha_N, \mathsf{Q}_1, \dots, \mathsf{Q}_N$ that belongs to each B of this form. Then $\mathsf{E}_{\mathsf{Q}_n} X \leq 0$ for any n and any $X \in A'$, which means that $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}$. Thus, the measure $\mathsf{Q} = \sum_n \alpha_n \mathsf{Q}_n$ belongs to $\operatorname{conv}_n(\mathcal{X}_n \cap \mathcal{R})$ and $\mathsf{E}_\mathsf{Q} F = x$.

Now, let $x = \mathsf{E}_\mathsf{Q} F$ with $\mathsf{Q} = \sum_n \alpha_n \mathsf{Q}_n$, $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}$. Suppose that there exist $X \in A$, $h \in \mathbb{R}$ such that, for $Y = X + h(F - x)$, we have $u_n(W_n + Y) > u_n(W_n)$ for each n. Due to the concavity of the function $\alpha \mapsto u_n(W_n + \alpha Y)$, we get

$$\begin{aligned} u_n(W_n + Y) - u_n(W_n) &\leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1}(u_n(W_n + \varepsilon Y) - u_n(W_n)) \\ &\leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1}\Big(\inf_{\mathsf{Q} \in \mathcal{X}_n} \mathsf{E}_\mathsf{Q}(W_n + \varepsilon Y) - u_n(W_n)\Big) = \inf_{\mathsf{Q} \in \mathcal{X}_n} \mathsf{E}_\mathsf{Q} Y. \end{aligned}$$

Consequently, $\mathsf{E}_{\mathsf{Q}_n} Y > 0$ for each n, and therefore, $\mathsf{E}_\mathsf{Q} Y > 0$. But, on the other hand, $\mathsf{Q} \in \mathcal{R}$, and therefore, $\mathsf{E}_\mathsf{Q} Y \leq \mathsf{E}_\mathsf{Q} h(F - x) = 0$. The contradiction shows that $x \in I_{NBC}(F)$. □

5 Equilibrium

5.1 Equilibrium in a Complete Model

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $u_1, \dots, u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1, \dots, \mathcal{D}_N$, let $A_1, \dots, A_N \subseteq L^0$ be convex sets containing zero such that A_n is $\mathcal{D}_n$-consistent for each n, and let $W_1 \in L^1_s(\mathcal{D}_1), \dots, W_N \in L^1_s(\mathcal{D}_N)$. From the financial point of view, u_n, A_n, and W_n are the coherent utility function, the "personal" set of attainable incomes, and the current endowment of the n-th agent, respectively.

Definition 5.1. A model is in *complete Pareto-type equilibrium* if there exists no collection $X_1, \dots, X_N, Y_1, \dots, Y_N$ such that

(a) $X_n \in A_n$ for each n;
(b) $\sum_{n=1}^N Y_n = 0$;
(c) $u_n(W_n + X_n + Y_n) \geq u_n(W_n)$ for each n and $u_n(W_n + X_n + Y_n) > u_n(W_n)$ for some n.

Remark. It follows from the translation invariance property ($u_n(Z + m) = u_n(Z) + m$) that condition (c) above can be replaced by each of the following conditions:

(c') $u_n(W_n + X_n + Y_n) > u_n(W_n)$ for each n;
(c") $\sum_{n=1}^N u_n(W_n + X_n + Y_n) > \sum_{n=1}^N u_n(W_n)$.

Definition 5.2. A model is in *complete Arrow-Debreu-type equilibrium* if there exists an *equilibrium price measure*, i.e. $\mathsf{Q} \in \mathcal{P}$ such that, for each n,

$$\max_{\substack{X \in A_n, \\ Y \in L^1_s(\mathsf{Q}) : \mathsf{E}_\mathsf{Q} Y = 0}} u_n(W_n + X + Y) = u_n(W_n).$$

The set of equilibrium price measures will be denoted by $\mathcal{E}$.

Theorem 5.3. *The following conditions are equivalent:*

(i) *a model is in complete Pareto-type equilibrium;*

(ii) *a model is in complete Arrow-Debreu-type equilibrium;*

(iii) $\bigcap_n(\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n))\neq\emptyset$.

Moreover, if these conditions are satisfied, then $\mathcal{E}=\bigcap_n(\mathcal{X}_{\mathcal{D}_n}(W_n)\cap\mathcal{R}(A_n))$.

Proof. Let us prove the implication (i)⇒(iii). Fix $M\in\mathbb{N}$ and collections $X_{nm}\in A'_n$, $n=1,\dots,N$, $m=1,\dots,M$; $Z_m\in L^\infty$, $m=1,\dots,M$. Consider the space

$$\mathbb{R}^{N\times M}\times\mathbb{R}^{N\times M}=\{(x,y):x=(x_{nm})_{n=1,\dots,N,\,m=1,\dots,M},y=(y_{nm})_{n=1,\dots,N,\,m=1,\dots,M}\}$$

and the sets

$$\begin{aligned}C=\{&(x,y)\in\mathbb{R}^{N\times M}\times\mathbb{R}^{N\times M}:\exists\mathsf{Q}_1\in\mathcal{X}_1,\dots,\mathsf{Q}_N\in\mathcal{X}_N:\\&x_{nm}=\mathsf{E}_{\mathsf{Q}_n}X_{nm},\ y_{nm}=\mathsf{E}_{\mathsf{Q}_n}Z_m,\ n=1,\dots,N,\ m=1,\dots,M\},\\K=\{&(x,y)\in\mathbb{R}^{N\times M}\times\mathbb{R}^{N\times M}:x_{nm}\le0,\ y_{1m}=\dots=y_{Nm},\\&n=1,\dots,N,\ m=1,\dots,M\},\end{aligned}$$

where $\mathcal{X}_n=\mathcal{X}_{\mathcal{D}_n}(W_n)$. Suppose that $C\cap K=\emptyset$. As C is a convex compact and K is a closed convex cone, there exist $\varepsilon>0$ and $(g,h)\in\mathbb{R}^{N\times M}\times\mathbb{R}^{N\times M}$ such that

$$\forall(x,y)\in C,\ \langle(g,h),(x,y)\rangle\ge\varepsilon, \tag{5.1}$$

$$\forall(x,y)\in K,\ \langle(g,h),(x,y)\rangle\le0. \tag{5.2}$$

Condition (5.2) means that $g_{nm}\ge0$ for any n,m and $\sum_{n=1}^N h_{nm}=0$ for any m. Consider $X_n=\sum_{m=1}^M g_{nm}X_{nm}$, $Y_n=\sum_{m=1}^M h_{nm}Z_m$. It follows from (5.1) that

$$\forall\mathsf{Q}_1\in\mathcal{X}_1,\dots,\mathsf{Q}_N\in\mathcal{X}_N,\ \sum_{n=1}^N\mathsf{E}_{\mathsf{Q}_n}(X_n+Y_n)\ge\varepsilon.$$

Set $b_n=\inf_{\mathsf{Q}\in\mathcal{X}_n}\mathsf{E}_\mathsf{Q}(X_n+Y_n)$ and consider $\widetilde{Y}_n=Y_n-b_n+\frac{1}{N}\sum_n b_n$. Then $\sum_n\widetilde{Y}_n=\sum_n Y_n=0$ and

$$\inf_{\mathsf{Q}\in\mathcal{X}_n}\mathsf{E}_\mathsf{Q}(X_n+\widetilde{Y}_n)=\frac{1}{N}\sum_{n=1}^N b_n\ge\frac{\varepsilon}{N}>0.$$

Employing now Theorem 1.15, we deduce that there exists a sufficiently small $\alpha>0$ such that $\alpha X_n\in A_n$ and $u_n(W_n+\alpha X_n+\alpha\widetilde{Y}_n)>u_n(W_n)$ for each n.

The obtained contradiction shows that, for any $M\in\mathbb{N}$, $X_{nm}\in A'_n$, $Z_m\in L^\infty$, the set

$$\begin{aligned}B(X_{nm},Z_m)=\Big\{&\mathsf{Q}_1,\dots,\mathsf{Q}_N\in\prod_{n=1}^N\mathcal{X}_n:\mathsf{E}_{\mathsf{Q}_n}X_{nm}\le0\text{ and}\\&\mathsf{E}_{\mathsf{Q}_1}Z_m=\dots=\mathsf{E}_{\mathsf{Q}_N}Z_m,\ n=1,\dots,N,\ m=1,\dots,M\Big\}\end{aligned}$$

is nonempty. As the map $\mathcal{X}_n\ni\mathsf{Q}\mapsto\mathsf{E}_\mathsf{Q}X$ is weakly continuous for each $X\in L^1_s(\mathcal{D}_n)$, the set $B(X_{nm},Z_m)$ is closed with respect to the product of weak topologies. Furthermore, any finite intersection of sets of this form is nonempty. Tikhonov's theorem, ensures that $\prod_n \mathcal{X}_n$ is compact. Consequently, there exists a collection $\mathsf{Q}_1, \dots, \mathsf{Q}_N \in \prod_n \mathcal{X}_n$ that belongs to each B of this form. Then $\mathsf{E}_{\mathsf{Q}_n} X \le 0$ for any $X \in A'_n$, which means that $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}(A_n)$. Furthermore, $\mathsf{E}_{\mathsf{Q}_1} Z = \dots = \mathsf{E}_{\mathsf{Q}_N} Z$ for any $Z \in L^\infty$, which means that $\mathsf{Q}_1 = \dots = \mathsf{Q}_N$.

Let us prove the implication (iii)$\Rightarrow$(i). Take $\mathsf{Q}_0 \in \bigcap_n (\mathcal{X}_n \cap \mathcal{R}(A_n))$. Suppose that there exist $X_1, \dots, X_N, Y_1, \dots, Y_N$ that satisfy conditions (a)–(c) of Definition 5.1. We have

$$\mathsf{E}_{\mathsf{Q}_0}(W_n + X_n + Y_n) \ge u_n(W_n + X_n + Y_n) > -\infty, \quad n = 1, \dots, N.$$

As $W_n \in L^1_s(\mathsf{Q}_0)$ and $\mathsf{E}_{\mathsf{Q}_0} X_n \le 0$, we conclude that $\mathsf{E}_{\mathsf{Q}_0} Y_n^- < \infty$, so that $\mathsf{E}_{\mathsf{Q}_0} Y_n \in (-\infty, \infty]$ for each n. As $\sum_n \mathsf{E}_{\mathsf{Q}_0} Y_n = \mathsf{E}_{\mathsf{Q}_0} \sum_n Y_n = 0$, we conclude that $Y_n \in L^1_s(\mathsf{Q}_0)$ for each n. As a result,

$$u_n(W_n + X_n + Y_n) \le \mathsf{E}_{\mathsf{Q}_0}(W_n + X_n + Y_n) \le \mathsf{E}_{\mathsf{Q}_0} W_n + \mathsf{E}_{\mathsf{Q}_0} Y_n = u_n(W_n) + \mathsf{E}_{\mathsf{Q}} Y_n, \quad n = 1, \dots, N.$$

But $\sum_n \mathsf{E}_{\mathsf{Q}_0} Y_n = 0$, so we get a contradiction.

In order to complete the proof, it is sufficient to verify the equality $\mathcal{E} = \bigcap_n (\mathcal{X}_n \cap \mathcal{R}(A_n))$. As each A_n contains zero and each function $\mathbb{R}_+ \ni \alpha \mapsto u_n(W_n + \alpha X + \alpha Y)$ is concave, a measure Q_0 belongs to $\mathcal{E}$ if and only if

$$\max_{\substack{X \in \operatorname{cone} A_n, \\ Y \in L^1_s(\mathsf{Q}_0) : \mathsf{E}_{\mathsf{Q}_0} Y = 0}} u_n(W_n + X + Y) = u_n(W_n), \quad n = 1, \dots, N.$$

Due to Theorem 3.7, this is equivalent to:

$$\inf_{\mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}(A_n + L)} \mathsf{E}_{\mathsf{Q}} W_n = \inf_{\mathsf{Q} \in \mathcal{D}_n} \mathsf{E}_{\mathsf{Q}} W_n, \quad n = 1, \dots, N,$$

where $L = \{Y \in L^1(\mathsf{Q}_0) : \mathsf{E}_{\mathsf{Q}_0} Y = 0\}$. Clearly, the latter condition is equivalent to: $\mathcal{X}_n \cap \mathcal{R}(A_n + L) \ne \emptyset$, $n = 1, \dots, N$. Obviously, this means that $\mathsf{Q}_0 \in \mathcal{X}_n \cap \mathcal{R}(A_n)$ for each n. □

Now, let $F \in L^0$ be the discounted payoff of a contingent claim.

Definition 5.4. A real number x is a *complete equilibrium price* of F if the extended model $\big(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}_n, A_n + \{h(F - x) : h \in \mathbb{R}\}, W_n; n = 1, \dots, N\big)$ is in complete equilibrium. The set of equilibrium prices will be denoted by $I_E(F)$.

Corollary 5.5. *For* $F \in \bigcap_n L^1_s(\mathcal{D}_n)$,

$$I_E(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{E}\}.$$

Proof. Denote $\{h(F - x) : h \in \mathbb{R}\}$ by $A(x)$. Clearly, for any n, $A_n + A(x)$ is $\mathcal{D}_n$-consistent. It follows from Theorem 5.3 that $x \in I_E(F)$ if and only if $\bigcap_n (\mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}(A_n + A(x))) \ne \emptyset$. It is easy to check that $\mathsf{Q} \in \mathcal{R}(A_n + A(x))$ if and only if $\mathsf{Q} \in \mathcal{R}(A_n)$ and $\mathsf{E}_{\mathsf{Q}} F = x$. This completes the proof. □

One can say that a model is in *complete equilibrium maximizing the overall utility* if

$$\sup_{\substack{X_1 \in A_1, \dots, X_N \in A_N, \\ Y_n \in L^0 : \sum_n Y_n = 0}} \sum_{n=1}^N u_n(W_n + X_n + Y_n) = \sum_{n=1}^N u_n(W_n).$$

Clearly, this notion of equilibrium is equivalent to the Pareto-type equilibrium.

Let us now assume that the system is not in equilibrium and find the maximal overall utility the agents can get by using their trading opportunities (i.e. $A_1, \dots, A_N$) as well as by exchanging arbitrary contracts.

Theorem 5.6. *If each A_n is a cone, then*

$$\sup_{\substack{X_1\in A_1,\dots,X_N\in A_N,\\ Y_n\in L^0:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n) = \inf_{\mathsf{Q}\in\bigcap_n(\mathcal{D}_n\cap\mathcal{R}(A_n))} \mathsf{E}_\mathsf{Q}\Bigl(\sum_{n=1}^N W_n\Bigr).$$

(If $u_n(W_n+X_n+Y_n)=-\infty$ for some n, we set $\sum_n u_n(W_n+X_n+Y_n)=-\infty$; we also use the convention $\inf\emptyset=\infty$.)

Lemma 5.7. *Let $u_1,\dots,u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1,\dots,\mathcal{D}_N$. Then, for any $X\in L^\infty$,*

$$\sup_{X_n\in L^\infty:\sum_n X_n=X} \sum_{n=1}^N u_n(X_n) = \inf_{\mathsf{Q}\in\bigcap_n\mathcal{D}_n} \mathsf{E}_\mathsf{Q}X. \tag{5.3}$$

(We use the convention $\inf\emptyset=\infty$.)

Remark. The left-hand side of (5.3) is called the *convex convolution* or the *inf-convolution* of $u_1,\dots,u_N$ (see [4], [16; Sect 5.2]). Thus, Lemma 5.7 states that it is a coherent utility function with the determining set $\bigcap_n\mathcal{D}_n$ if $\bigcap_n\mathcal{D}_n\neq\emptyset$ and it is identically equal to $+\infty$ if $\bigcap_n\mathcal{D}_n=\emptyset$.

Proof of Lemma 5.7. In the case, where $\bigcap_n\mathcal{D}_n\neq\emptyset$, this statement follows by induction from a result proved in [16; Sect 5.2].

Assume now that $\bigcap_n\mathcal{D}_n=\emptyset$. Find m such that $\bigcap_{n=1}^m\mathcal{D}_n\neq\emptyset$, while $\bigcap_{n=1}^{m+1}\mathcal{D}_n=\emptyset$. By the Hahn-Banach theorem, there exists $Z\in L^\infty$ such that

$$\sup_{\mathsf{Q}\in\mathcal{D}_{m+1}} \mathsf{E}_\mathsf{Q}Z<0<\inf_{\mathsf{Q}\in\bigcap_{n=1}^m\mathcal{D}_n} \mathsf{E}_\mathsf{Q}Z.$$

According to the part of the lemma that has already been proved, there exist $Z_1,\dots,Z_m\in L^\infty$ such that $\sum_{n=1}^m Z_n=Z$ and $\sum_{n=1}^m u_n(Z_n)>0$. Then $u_1(Z_1)+\dots+u_m(Z_m)+u_{m+1}(-Z)>0$. Consequently, the left-hand side of (5.3) is identically equal to ∞. □

Proof of Theorem 5.6. Employing the same arguments as those used in the proof of the implication (iii)⇒(i) of Theorem 5.3, we conclude that

$$\sup_{\substack{X_1\in A_1,\dots,X_N\in A_N,\\ Y_n\in L^0:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n) \le \inf_{\mathsf{Q}\in\bigcap_n(\mathcal{D}_n\cap\mathcal{R}(A_n))} \mathsf{E}_\mathsf{Q}\Bigl(\sum_{n=1}^N W_n\Bigr). \tag{5.4}$$

Let us prove the reverse inequality. Clearly, it is sufficient to prove it for bounded W_n. It follows from Theorem 3.7 that

$$\sup_{\substack{X_1\in A_1,\dots,X_N\in A_N,\\ Y_n\in L^\infty:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n) = \sup_{Y_n\in L^\infty:\sum_n Y_n=0} \sum_{n=1}^N \inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}(A_n)} \mathsf{E}_\mathsf{Q}(W_n+Y_n)$$

(note that here we take $Y_n\in L^\infty$). Combining this equality with Lemma 5.7, we get

$$\sup_{\substack{X_1\in A_1,\dots,X_N\in A_N,\\ Y_n\in L^\infty:\sum_n Y_n=0}} \sum_{n=1}^N u_n(W_n+X_n+Y_n) = \inf_{\mathsf{Q}\in\bigcap_n(\mathcal{D}_n\cap\mathcal{R}(A_n))} \mathsf{E}_\mathsf{Q}\Bigl(\sum_{n=1}^N W_n\Bigr). \tag{5.5}$$

Inequality (5.4) and equality (5.5) combined together yield the desired statement. □

5.2 Equilibrium in an Incomplete Model

Let u_n, $\mathcal{D}_n$, A_n, and W_n be the same as in the previous subsection. Let $S_1^1, \dots, S_1^d \in \bigcap_n L_s^1(\mathcal{D}_n)$ be the discounted prices at time 1 of several financial assets. (There is no relation between S_1 and A_n; A_n means the set of incomes that can be obtained by the n-th agent without trading the assets $1, \dots, d$.)

Definition 5.8. A model is in *incomplete Pareto-type equilibrium* if there exists no collection $X_1, \dots, X_N, Y_1, \dots, Y_N$ such that

(a) $X_n \in A_n$ for each n;

(b) $Y_n \in \operatorname{span}(1, S_1^1, \dots, S_1^d)$ and $\sum_{n=1}^N Y_n = 0$;

(c) $u_n(W_n+X_n+Y_n) \geq u_n(W_n)$ for each n and $u_n(W_n+X_n+Y_n) > u_n(W_n)$ for some n.

Definition 5.9. A model is in *incomplete Arrow-Debreu-type equilibrium* if there exists an *equilibrium price vector*, i.e. $S_0 \in \mathbb{R}^d$ such that, for each n,

$$\max_{X \in A_n,\, h \in \mathbb{R}^d} u_n(W_n + X + \langle h, S_1 - S_0 \rangle) = u_n(W_n). \tag{5.6}$$

The set of equilibrium price vectors will be denoted by E.

Let us introduce the notation

$$C_n = \{\mathsf{E}_{\mathsf{Q}} S_1 : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}(A_n)\}.$$

Theorem 5.10. *The following conditions are equivalent:*

(i) *a model is in incomplete Pareto-type equilibrium;*

(ii) *a model is in incomplete Arrow-Debreu-type equilibrium;*

(iii) $\bigcap_n C_n \neq \emptyset$.

Moreover, if these conditions are satisfied, then $E = \bigcap_n C_n$.

Proof. The implication (i)$\Rightarrow$(iii) is proved in the same way as in Theorem 5.3. The only difference is that instead of taking Z_m from L^∞, one should take Z_m from $\operatorname{span}(S_1^1, \dots, S_1^d)$.

Let us prove the implication (iii)$\Rightarrow$(i). Let $S_0 \in \bigcap_n C_n$, i.e. there exist $\mathsf{Q}_n \in \mathcal{X}_n \cap \mathcal{R}(A_n)$ such that $S_0 = \mathsf{E}_{\mathsf{Q}_n} S_1$, $n = 1, \dots, N$. Suppose that there exist $X_1, \dots, X_N, Y_1, \dots, Y_N$ that satisfy conditions (a)–(c) of Definition 5.8. We can write $Y_n = a_n + \langle h_n, S_1 - S_0 \rangle$ with some $a_n \in \mathbb{R}$, $h_n \in \mathbb{R}^d$. It follows from the inequality

$$\begin{aligned} u_n(W_n + X_n + Y_n) &\leq \mathsf{E}_{\mathsf{Q}_n}(W_n + X_n + a_n + \langle h_n, S_1 - S_0 \rangle) \\ &\leq \mathsf{E}_{\mathsf{Q}_n} W_n + a_n = u_n(W_n) + a_n \end{aligned} \tag{5.7}$$

that $a_n \geq 0$ for each n. But, on the other hand, $\sum_n a_n = \mathsf{E}_{\mathsf{Q}_1} \sum_n Y_n = 0$. Consequently, each a_n is zero. Employing (5.7), we deduce that there exists no n, for which $u_n(W_n + X_n + Y_n) > u_n(W_n)$.

In order to complete the proof, it is sufficient to verify the equality $E = \bigcap_n C_n$. This is done by the same arguments as those used in the proof of Theorem 4.8. □

Definition 5.11. An *incomplete equilibrium price* of a contingent claim F is an equilibrium price vector corresponding to $S_1 = F$.

The set of equilibrium prices will be denoted by $I_E(F)$.

Corollary 5.12. *For* $F \in L_s^1(\mathcal{D})$,

$$I_E(F) = \bigcap_{n=1}^N \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{X}_{\mathcal{D}_n}(W_n) \cap \mathcal{R}(A_n)\}.$$

Remark. In view of Theorem 4.8, $I_E(F) = \bigcap_n I_{NBC(\mathcal{D}_n, A_n, W_n)}(F)$.

5.3 Demand-Supply Equilibrium Pricing

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $u_1, \dots, u_N$ be coherent utility functions with the weakly compact determining sets $\mathcal{D}_1, \dots, \mathcal{D}_N$, let $A_1, \dots, A_N \subseteq L^0$ be convex cones such that A_n is $\mathcal{D}_n$-consistent for each n, and let $W_1 \in L_s^1(\mathcal{D}_1), \dots, W_N \in L_s^1(\mathcal{D}_N)$.

Let $F \in \bigcap_n L_s^1(\mathcal{D}_n)$ be a contingent claim. Note that if x is an incomplete equilibrium price of F, then, in view of (5.6), the optimal amount of F in the portfolio of each agent is zero. The same property is true if x is a complete equilibrium price because (due to Corollaries 5.5 and 5.12) the set of complete equilibrium prices belongs to the set of incomplete equilibrium prices. Thus, in practice the equilibrium price intervals introduced in Subsections 5.1 and 5.2 would typically be empty. Let us now give a more realistic definition.

Definition 5.13. A *demand-supply equilibrium price* of a contingent claim F is a real number x, for which there exist $h_{1*}, \dots, h_{N*} \in \mathbb{R}$ such that $\sum_n h_{n*} = 0$ and, for any n,

$$\sup_{X \in A_n,\, h \in \mathbb{R}} u_n(W_n + X + h(F - x)) = \sup_{X \in A_n} u_n(W_n + X + h_{n*}(F - x)). \tag{5.8}$$

The set of equilibrium prices will be denoted by $I_E(F)$.

`Remarks.` `(i)` A number h_{n*} satisfying (5.8) means the optimal amount of the contract F in the portfolio of the n-th agent.

`(ii)` The introduction of a new contract F would shift the prices of the existing contracts, i.e. would change A_n. However, we assume that this effect is small and use in Definition 5.13 the same A_n as those in the original model.

`(iii)` Suppose that in addition to h_{n*} there exist $X_{1*} \in A_1, \dots, X_{N*} \in A_N$ such that, for any n,

$$\max_{X \in A_n,\, h \in \mathbb{R}} u_n(W_n + X + h(F - x)) = u_n(W_n + X_{n*} + h_{n*}(F - x)).$$

Then the model $(\Omega, \mathcal{F}, \mathsf{P}, \mathcal{D}_n, A_n, W_n + X_{n*} + h_{n*}(F - x); n = 1, \dots, N)$ is in equilibrium in the sense of Definitions 5.8, 5.9 (with $S_1 = F$).

`(iv)` Let $F \in \bigcap_n L_s^1(\mathcal{D}_n)$ and let $I_1(F)$, $I_2(F)$, and $I_3(F)$ denote the complete, the incomplete, and the demand-supply equilibrium price intervals, respectively. Then $I_1(F) \subseteq I_2(F) \subseteq I_3(F)$. Indeed, the first inclusion follows from Corollaries 5.5 and 5.12. Furthermore, if $x \in I_2(F)$, then, in view of (5.6), condition (5.8) is satisfied with $h_{n*} = 0$.

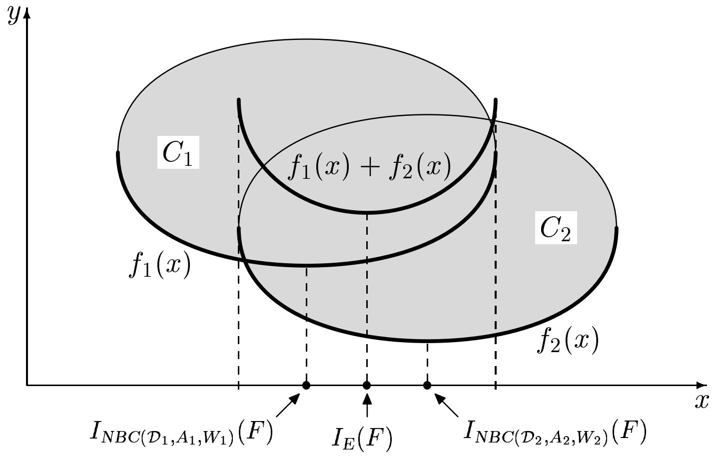
Let us introduce the notation (see Figure 12, compare with Figure 11)

$$\begin{aligned} I_n &= \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}(A_n)\}, \\ C_n &= \{\mathsf{E}_\mathsf{Q}(F, W_n) : \mathsf{Q} \in \mathcal{D}_n \cap \mathcal{R}(A_n)\}, \\ f_n(x) &= \inf\{y : (x, y) \in C_n\}, \quad x \in I_n \end{aligned}$$

and let I_n° denote the interior of I_n.

Theorem 5.14. *Assume that* $\bigcap I_n^\circ \neq \emptyset$. *Then, for* $F \in \bigcap_n L_s^1(\mathcal{D}_n)$,

$$I_E(F) = \operatorname*{argmin}_{x \in \bigcap_n I_n} \sum_{n=1}^N f_n(x).$$



`Figure 12.` The form of the demand-supply equilibrium price interval. In this example, $I_{NBC(\mathcal{D}_1,A_1,W_1)}(F)\cap I_{NBC(\mathcal{D}_2,A_2,W_2)}(F)=\emptyset$, so that both complete and incomplete equilibrium price intervals are empty.

`Proof`. Fix $n\in\{1,\dots,N\}$ and $x\in\bigcap_n I_n$. By Theorem 3.7, for any fixed h, we have

$$\sup_{X\in A} u_n(W_n+X+h(F-x))=\inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}(A_n)}\mathsf{E_Q}(W_n+h(F-x)).$$

Thus, a number h_{n*} satisfies (5.8) if and only if

$$h_{n*}\in\operatorname*{argmax}_{h\in\mathbb{R}}\inf_{\mathsf{Q}\in\mathcal{D}_n\cap\mathcal{R}(A_n)}\mathsf{E_Q}(W_n+h(F-x)).$$

It is easy to see from Theorem 3.8 that the latter condition is satisfied if and only if the vector $(h_{n*},1)$ is an inner normal to C_n at the point $(x,f_n(x))$. This, in turn, is equivalent to the inclusion $h_{n*}\in g_n(x)$, where

$$g_n(x)=\begin{cases}[(f_n)'_-(x),(f_n)'_+(x)] & \text{if } x\in I_n^\circ,\\ (-\infty,(f_n)'_+(x)] & \text{if } x \text{ is the left endpoint of } I_n,\\ [(f_n)'_-(x),\infty) & \text{if } x \text{ is the right endpoint of } I_n.\end{cases}$$

Here $(f_n)'_-$ (resp., $(f_n)'_+$) denotes the left-hand (resp., the right-hand) derivative of f_n. It follows from the condition $\bigcap_n I_n^\circ\neq\emptyset$ that $x\in\operatorname{argmin}_{x\in\bigcap_n I_n}\sum_n f_n(x)$ if and only if $\sum_n g_n(x)\ni 0$. This completes the proof. □

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