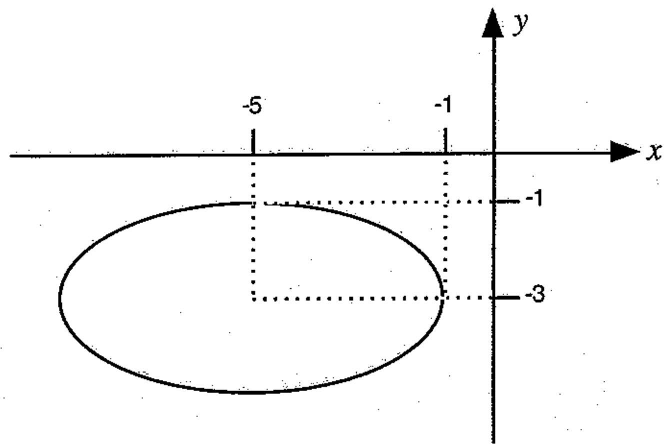
SOLUTIONS

Math 259 Winter 2009

Recitation Handout 1: Finding Formulas for Parametric Curves

1. The diagram given below shows an ellipse in the xy-plane.



(a) Find equations for $x(t)$ and $y(t)$ that will describe this curve. Use the interval $0 \le t \le 2\pi$.

$$x(t) = -5 + 4 \cdot \cos(t)$$
$$y(t) = -3 + 2 \cdot \sin(t)$$

Other solutions are possible. This curve begins at the point $(-1,-3)$ and is traced out in a counterclockwise direction.

(b) How would you modify your answer to Part (a) if only the right hand half of the ellipse was needed?

Change the interval of t-values from $[0, 2\pi]$ to $[-\pi/2, \pi/2]$.

(c) How would you modify your answer to Part (b) if the initial and terminal points of the curve are reversed?

If we want to use the interval of t-values $[-\pi/2, \pi/2]$ then we can change the equations to:

$$x(t) = -5 + 4 \cdot \cos(t)$$
$$y(t) = -3 - 2 \cdot \sin(t)$$

(other approaches are possible.)

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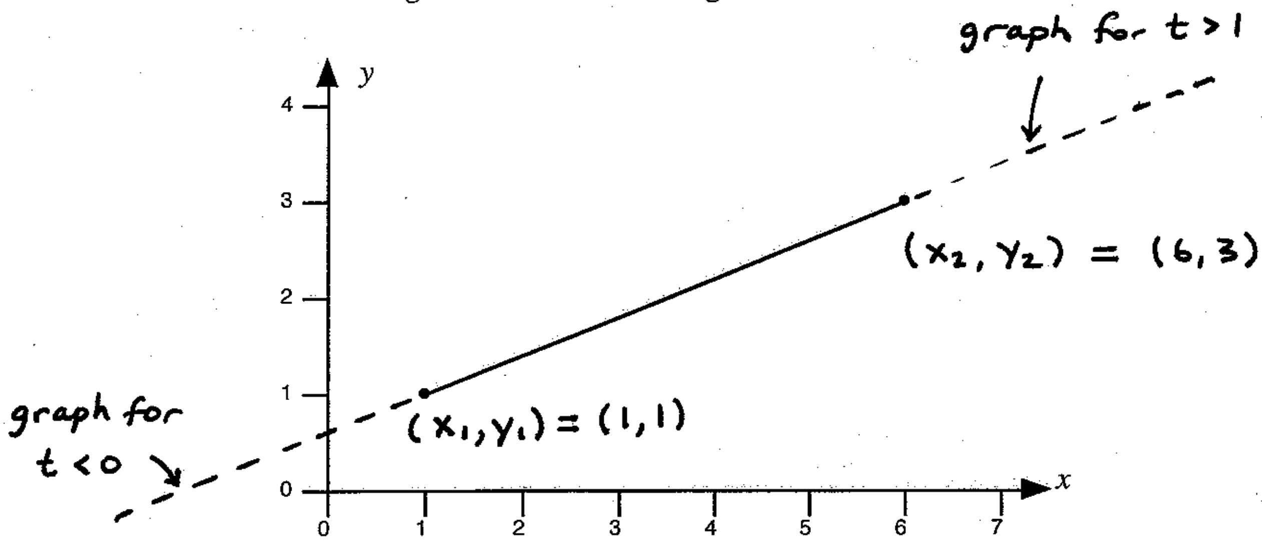
2. Consider the line segment shown in the diagram below.



(a) Find a pair of parametric equation *and* an interval of t-values that correspond to the line segment pictured above, and that line segment only.

$$x(t) = 1\cdot(1-t) + 6\cdot t = 1+5t$$

$$y(t) = 1\cdot(1-t) + 3\cdot t = 1+2t$$

Interval: $0 \le t \le 1.$

(other solutions are possible.)

(b) Suppose the lowest t-value in your interval is $t = a$. Sketch the graph you would get if you were to plug values of t that are less than $t = a$ into your parametric equations from Part (a). The graph is a continuation of the above line. It extends to the left of the line segment.

(c) Suppose the highest t-value in your interval is $t = b$. Sketch the graph you would get if you were to plug values of t that are less than $t = b$ into your parametric equations from Part (a). The graph is a continuation of the above line. It extends to the right of the line segment.

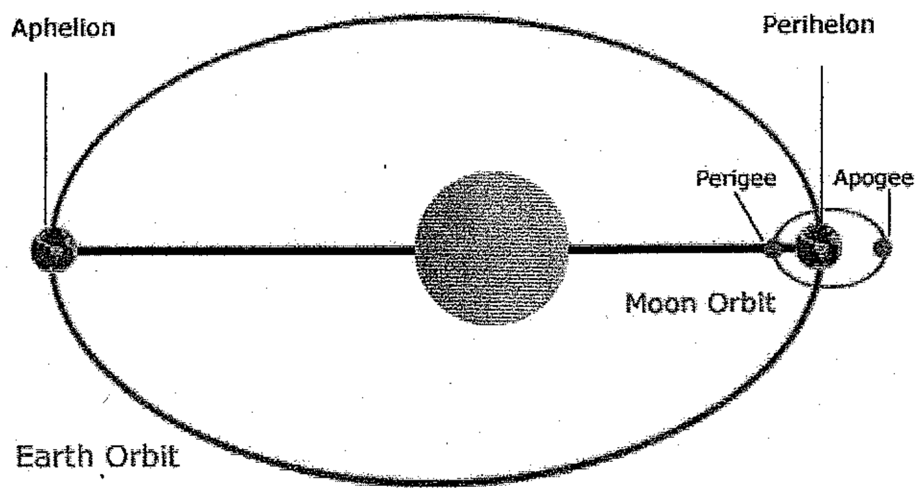
(d) Find a pair of parametric equations that specify the straight line that passes through the point (1, 1) and is perpendicular to the line segment pictured above.

Two points on the perpendicular line are (1, 1) and (−1, 6). The parametric equations for the line going through these points are:

$$x(t) = 1\cdot(1-t) + -1\cdot t = 1-2t$$

$$y(t) = 1\cdot(1-t) + 6t = 1+5t.$$

3. The Earth orbits the sun with an orbital path that is actually an ellipse, but which is so close to being a circle that we will assume that the orbital path is circular. The Moon orbits the Earth in an orbit that is also very close to being circular (so we will assume it to be a circle also), as shown in the diagram[1] given below.



Throughout this problem we will assume that t represents the number of days after midnight on January 1.

(a) The mean distance from the Earth to the Moon is 384,400 km. Suppose that at time $t = 0$ the Moon starts at the point labeled "Apogee" in the above diagram and starts to orbit the Earth in a counterclockwise direction. Imagine that the Earth is fixed at the point (0, 0) of the xy-plane. Find a pair of parametric equations to describe the motion of the Moon around the Earth.

Note that it takes approximately 28 days for the Moon to go around the Earth once.

$$x(t) = 384\,400 \cdot \cos\left(\frac{2\pi}{28} t\right)$$

$$y(t) = 384\,400 \cdot \sin\left(\frac{2\pi}{28} t\right).$$

[1] Image courtesy of NASA, http://www.nasa.gov

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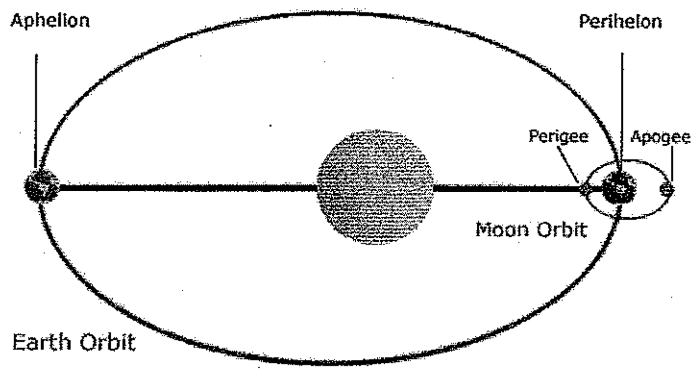

(b) The mean distance from the Earth to the Sun is 149,600,000 km. Suppose that at time $t = 0$ the Earth starts at the point labeled "Perihelon" in the diagram (above) and immediately starts to orbit the Sun in a counterclockwise direction. Imagine now that the Sun is fixed at the point (0, 0) of the xy-plane. Find a pair of parametric equations to describe the motion of the Earth around the Sun.

Note that it takes approximately 365 days for the Earth to go around the Sun once.

$$x(t) = 149600000 \cdot \cos\left(\frac{2\pi}{365} t\right)$$

$$y(t) = 149600000 \cdot \sin\left(\frac{2\pi}{365} t\right)$$

(c) Imagine again that the Sun is fixed at the point (0, 0) of the xy-plane and that at time $t = 0$ the Moon starts at the point labeled "Apogee" on the above diagram. Bearing in mind that the Moon orbits the Earth and the Earth orbits the Sun, find a pair of parametric equations to describe the motion of the Moon around the Sun.

We can just add the two sets of parametric equations to get the combined motion.

$$x(t) = 149600000 \cdot \cos\left(\frac{2\pi}{365} t\right) + 384400 \cdot \cos\left(\frac{2\pi}{28} t\right)$$

$$y(t) = 149600000 \cdot \sin\left(\frac{2\pi}{365} t\right) + 384400 \cdot \sin\left(\frac{2\pi}{28} t\right).$$

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4. In a search and rescue mission, a Coast Guard cutter is searching for a yacht that is lost in the fog and in a state of distress. Imagine that the area of ocean that the fog covers is like the xy-plane. Let t be the number of hours that have passed since the yacht was reported overdue and the Coast Guard search began.

The position of the yacht at time t is given by the pair of parametric equations:

$$x_Y(t) = 4t - 4 \qquad \text{and} \qquad y_Y(t) = 2t - k,$$

where k is a constant. The position of the Coast Guard cutter at time t is given by the pair of parametric equations:

$$x_{CG}(t) = 3t \qquad \text{and} \qquad y_{CG}(t) = t^2 - 2t - 1.$$

(a) If it is determined that $k = 5$, does the Coast Guard ever find the yacht?

To find t, solve $x_Y(t) = x_{CG}(t)$. $3t = 4t - 4$ $\boxed{t = 4}$

Plug $t = 4$ into $y_Y(t)$ to get: $y_Y(4) = 8 - 5 = 3$

Plug $t = 4$ into $y_{CG}(t)$ to get: $y_{CG}(4) = 16 - 8 - 1 = 7$.

Since $y_Y(4) \neq y_{CG}(4)$ the two boats never come together.

(b) Find the values of k that result in the Coast Guard eventually finding the yacht.

Know that $x_Y(t) = x_{CG}(t)$ only when $t = 4$.

Find value(s) of k that solve $y_Y(4) = y_{CG}(4)$.

$8 - k = 7$ so $\boxed{k = 1}$.

(c) At the time that the Coast Guard finds the yacht in Part (b), which vessel is sailing the fastest? (Your answer may contain the constant k.)

Yacht: Speed $= \sqrt{x_Y'(4)^2 + y_Y'(4)^2}$

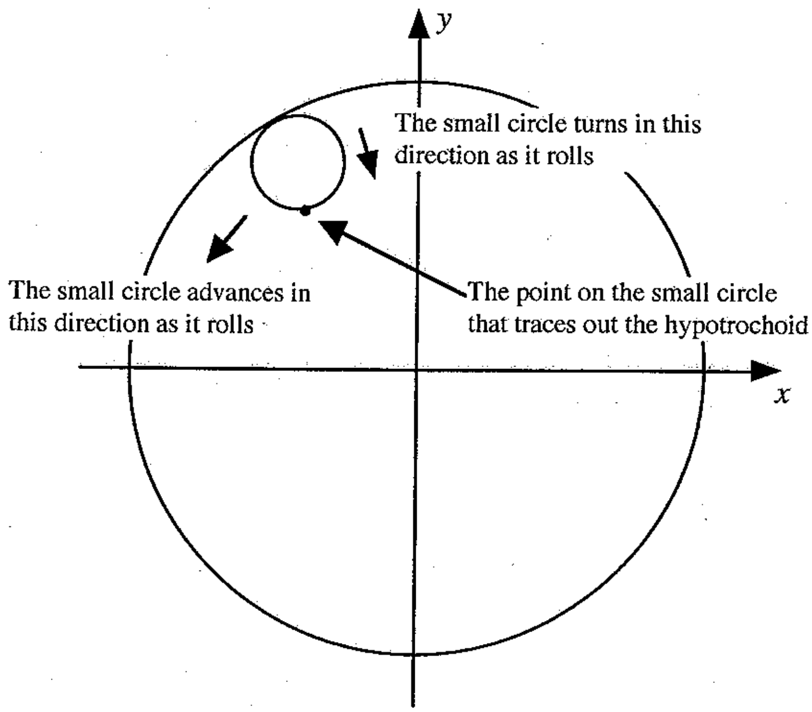
$= \sqrt{16 + 4} = \sqrt{20}$

Coast Guard: Speed $= \sqrt{x_{CG}'(4)^2 + y_{CG}'(4)^2}$

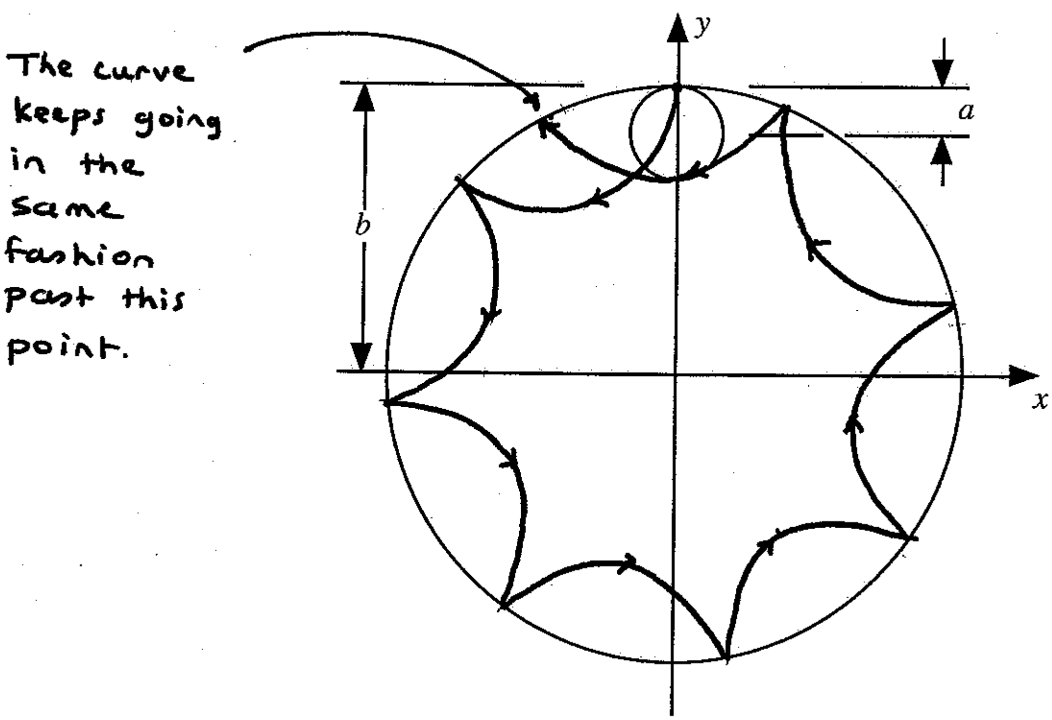
$= \sqrt{9 + 36} = \sqrt{45}$

The Coast Guard cutter is sailing the fastest.

5. A **hypotrochoid** is a curve formed when a circle rolls around inside another circle, like the ball in a game of roulette rolling around the track of the roulette wheel. Formally, imagine a point on the edge of a circle with radius a (with $0 < a$) that rolls around the inside of another circle of radius b (with $0 < a < b$) as shown below. The curve traced out by the point on the smaller circle as the small circle rolls around is the hypotrochoid.



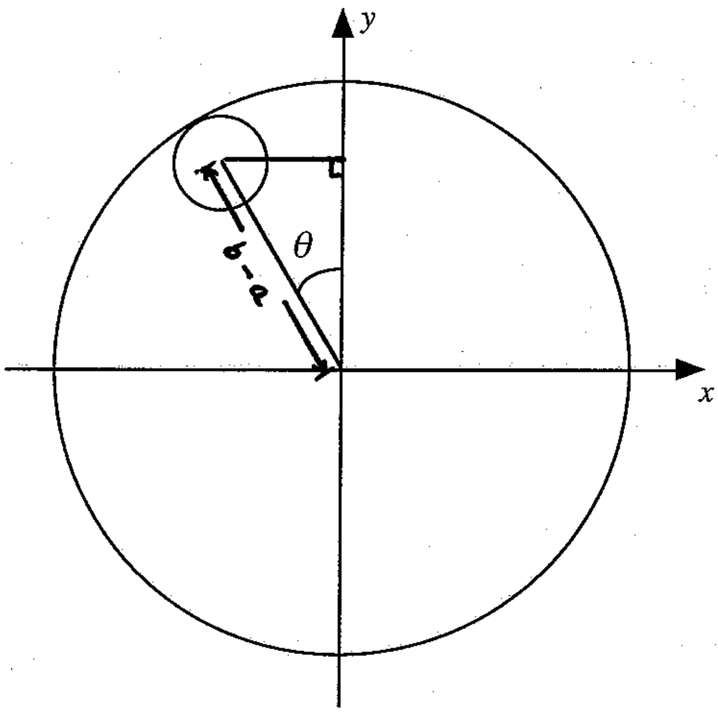
(a) Use the diagram provided below to sketch the hypotrochoid.



NOTE: This is a sketch. Your sketch should be similar. It does not have to be identical.

(b) Find a pair of parametric equations ($x(\theta)$ and $y(\theta)$) that describe the hypotrochoid. It is fine for your equations to include the constants a and b.

NOTE: The suggested parameter, θ, is the angle between the line segment joining the centers of the two circles and the y-axis, as shown in the diagram below. If you believe that another parameter will be easier to use, feel free to use it – but make sure you define the alternative parameter clearly when you write down your solution!



The coordinates of the center of the small circle are given by:

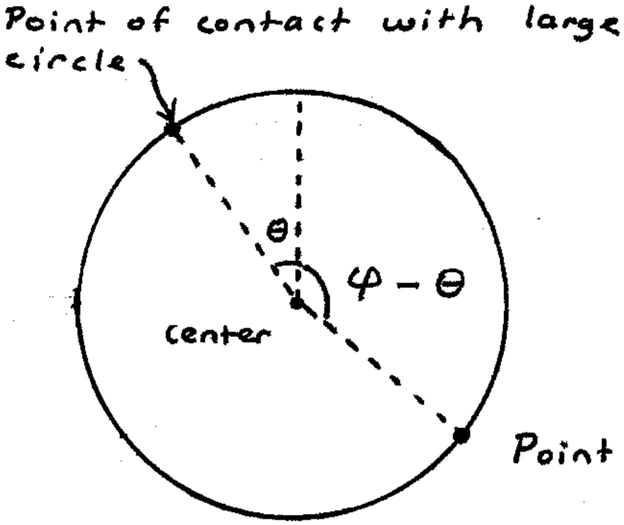
$$x = (b-a)\cdot(-\sin\theta)$$
$$= (a-b)\cdot\sin(\theta)$$

$$y = (b-a)\cdot\cos(\theta)$$

As the small circle rolls, the arc length that it covers is $b\cdot\theta$. If the angle that the small circle has turned around its own center is φ then:

$$a\cdot\varphi = b\cdot\theta$$

so $\varphi = \frac{b}{a}\cdot\theta$. The next diagram shows the small circle and the marked point on its edge. (See next page.) The distances between the center of the small circle and the marked point in both the x and y directions are given by:



- x-direction:

$$\text{Distance} = a \cdot \sin(\varphi - \theta)$$
$$= a \cdot \sin\left(\frac{b-a}{a} \cdot \theta\right)$$

- y-direction:

$$\text{Distance} = a \cdot \cos(\varphi - \theta)$$
$$= a \cdot \cos\left(\frac{b-a}{a} \cdot \theta\right)$$

Adding these distances to the x and y coordinates of the center of the small circle gives the parametric equations:

$$x(\theta) = (a-b) \cdot \sin(\theta) + a \cdot \sin\left(\frac{b-a}{a} \cdot \theta\right)$$

$$y(\theta) = (b-a) \cdot \cos(\theta) + a \cdot \cos\left(\frac{b-a}{a} \cdot \theta\right)$$