Handout 6: In-class Review for the Cumulative Final Exam

The topics covered by the cumulative final exam include the following:

- Parametric curves.
- Finding formulas for parametric curves.
- Drawing graphs of curves defined by parametric equations.
- Finding tangent lines to curves defined by parametric equations.
- Finding the area beneath (between the curve and the x-axis) a parametric curve.
- Finding the arc length of a parametric curve.
- Polar coordinates for the xy-plane.
- Identifying regions of the xy-plane described by polar coordinates.
- Converting Cartesian equations to polar equations.
- Converting polar equations to Cartesian equations.
- Sketching curves in the xy-plane defined by polar equations.
- Finding formulas for tangent lines to curves defined by polar equations.
- Finding areas enclosed by polar curves.
- Finding arc lengths of curves defined by polar equations.
- Conic sections in Cartesian and polar coordinates.
- Sketching conic sections defined by polar equations. Identifying eccentricity, directrix, etc. from a polar equation. Classifying conic sections using eccentricity.
- Equations of lines, planes and spheres in 3D.
- Combining vectors. Magnitude of a vector. Unit vectors.
- Applications of vectors in physics.
- Dot product of vectors. Angle between vectors. Orthogonality. Vector projections.
- Cross product of vectors. Geometry of the cross product. Cross product and areas.
- Calculating volumes with the scalar triple product.
- Finding equations for lines and planes in 3D using the cross product.
- Distances from points to lines and planes, and from lines to planes.
- Symmetric equations.
- Sketching surfaces in 3D using contour plots.
- Interpreting contour plots.
- Classifying quadric surfaces.
- Recognizing planes, quadric surfaces and cylinders from their equations.
- Creating and using vector functions in 3D.
- Velocities and tangent vectors for vector functions (including unit tangent vectors).
- Showing that limits in 2D do not exist using a variety of strategies (e.g. $y = mx$, contour plots or tables).
- Evaluating and using functions with several input variables.
- Proving that limits do exist using the ε-δ definition.
- Calculating values for limits/showing limits exist using the Squeezing Theorem.
- Calculating and interpreting partial derivatives.
- Finding equations for tangent planes.
- Using the tangent plane to calculate a linear approximation.
- Calculating total differentials for functions.
- Using total differentials to estimate changes and errors.
- Using the Chain Rule for functions of several variables.
- Calculating directional derivatives for functions.
- Calculating gradient vectors.
- Finding the direction of maximum rate of change (and magnitude of the maximum rate of change).
- Interpreting the practical meaning of a directional derivative.
- Finding and classifying (local maximum, local minimum, saddle point) the critical points of a function of several variables using partial derivatives and the Jacobian determinant.
- Finding the global maximum and global minimum of a continuous function over a region in the xy-plane.
- Lagrange Multipliers.
- Lagrange Multipliers with two constraints.
- Word problems that can be solved using Lagrange Multipliers.

- Double Riemann sums.
- The Midpoint Rule for double Riemann sums (no error estimates).
- Setting up double integrals.
- Changing the order of integration in a double integral.
- Evaluating double integrals.
- Applications of double integrals to chemistry and physics (e.g. volume, mass, center of mass).
- Polar coordinates.
- Converting double integrals from Cartesian to polar coordinates.
- Setting up double integrals in polar coordinates.
- Evaluating double integrals in polar coordinates.
- Setting up triple integrals.
- Changing the order of integration for a triple integral.
- Evaluating triple integrals.
- Applications of triple integrals (e.g. calculating volume and mass).
- Setting up and evaluating triple integrals in cylindrical/polar coordinates.
- Setting up and evaluating triple integrals in spherical coordinates.
- Drawing two-dimensional vector fields.
- Associating formulas for two-dimensional vector fields with visual representations.
- Determining whether a vector field is conservative or not.
- Finding potential functions for conservative vector fields.
- Setting up parametric equations for curves in two dimensions.
- Setting up and evaluating (directly) line integrals of functions in two dimensions.
- Setting up and evaluating (directly) line integrals of vector fields in two and three dimensions.
- Calculating line integrals of conservative vector fields using the Fundamental Theorem.
- Calculating line integrals in two dimensions using Green's Theorem.
- Divergence and curl operators.
- Setting up and evaluating surface integrals.
- Evaluating surface integrals using Stoke's Theorem and the Divergence Theorem.

1. Find symmetric equations for the line of intersection of the planes:

$$x - y + z = 5$$

$$2x + y - 3z = 4.$$

2. Find the areas of the planar (flat) shapes that have vertices located at:

(a) (0, 0), (4, 1), (2, 3), and (6, 4).

(b) (2, -3), (1, 1), (5, -6), and (4, -2).

(c) (0, 0, 0), (0, 1, 0), and (1, 1, 0).

(d) (1, 3, 0), (0, 2, 5), and (-1, 0, 2).

3. **(a)** Determine values of λ and μ so that the points (-1, 3, 2), (-4, 2, -2) and (5, λ, μ) lie on a straight line.

(b) Find a value of λ that will make the three vectors:

$\mathbf{a} = \mathbf{I} + \mathbf{j} + \mathbf{k}$ $\mathbf{b} = 2\mathbf{I} - 4\mathbf{k}$ $\mathbf{c} = \mathbf{I} + \lambda\mathbf{j} + 3\mathbf{k}$

coplanar.

(c) Suppose that the pairs of vectors **A** and **B**, and **C** and **D** each determine a plane. Show that if these planes are parallel, then: $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = 0$.

4. Consider the curve in the plane defined by the parametric equations:

$$x(t) = \sin(t) \qquad y(t) = \sin(2t) \qquad 0 \le t \le \pi.$$

(a) Sketch the graph of this curve in the *xy* plane.

(b) Find the total length of the curve.

(c) Find the area enclosed by this curve.

5. In this problem we will consider the two lines given by:

$$x = 1 - t,\ y = -2 + \text{t},\ z = 2t \qquad \text{and} \qquad x = 4 + s,\ y = 1 + 3s,\ z = -3 + s.$$

(a) Determine whether or not these lines are parallel.

(b) If the lines are not parallel, do they intersect? If so, find the coordinates of the point of intersection.

(c) Find the minimum distance between the two lines?

6. In each of the following cases, find the volume of the parallelpiped that has vertices at the points listed.

(a) (1, 1, 1), (-4, 2, 7), (3, 5, 7) and (0, 1, 6).

(b) (1, 6, 1), (-2, 4, 2), (3, 0, 0), and (2, 2, -4).

(c) (1, 1, 1), (2, 2, 2), (6, 1, 3), and (-2, 4, 6).

7. An *astroid* (not to be confused with an *asteroid*) is a mathematical curve with the equation,

$$x^{2/3} + y^{2/3} = 1.$$

This curve can be described by the parametric equations:

$$x = \cos^3(t) \qquad \text{and} \qquad y = \sin^3(t) \qquad \text{with} \qquad 0 \le t \le 2\pi.$$

(a) Sketch a graph of the astroid.

(b) Find a formula for $\frac{dy}{dx}$ the astroid.

(c) What is the total length of the astroid?

(d) Set up an integral that gives the area enclosed by the astroid.

8. The position vector of a particle moving in the xy plane is given below.

$$\mathbf{r}(t) = e^t\sin(t)\mathbf{i} + e^t\cos(t)\mathbf{j}$$

(a) Sketch the path of the particle beginning with $t = 0$.

(b) Find the tangential component of the particle's acceleration. (That is, the vector projection of the acceleration vector onto the velocity vector.)

(c) Find the normal component of the particle's acceleration. (That is, the component of the acceleration vector that is normal to the velocity vector.)

(d) How could you deduce from the sketch in part (a) that the normal component of the acceleration was going to be non-zero?

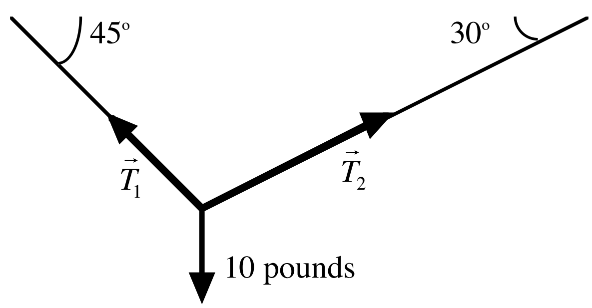
(e) Describe the motion of a particle that would have a zero normal component of acceleration.

9. **(a)** Find an equation for the plane that includes both the point (1, 3, 0) and the line

$$\langle x, y, z \rangle = \langle 2, 7, 1 \rangle + t \cdot \langle -1, 1, 1 \rangle.$$

Express your final answer in the form: $ax + by + cz = d$.

(b) The diagram given below shows a weight of 10 pounds suspended from a cord. Calculate the tension vector, $\vec{T}_2$, in the cord to the **right** of the suspended weight. Calculate the *vector*, not just the magnitude of the vector.



10. In this problem $\vec{u}$, $\vec{v}$ and $\vec{w}$ will always refer to the following vectors:

$\vec{u}$ = <1, 2, −3> $\vec{v}$ = <−7, −14, 21> $\vec{w}$ = <0, 1, 1>.

(a) Calculate $\vec{v} \times \vec{w}$.

(b) Calculate the volume of the parallelpiped whose sides are formed by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

(c) Calculate $\vec{u} \times \vec{v}$.

(d) What can you conclude about the vectors $\vec{u}$ and $\vec{v}$? Circle any of the statements that you believe to be true.

i. $\vec{u}$ and $\vec{v}$ are parallel (or anti-parallel).

ii. $\vec{u}$ and $\vec{v}$ are both perpendicular to the vector $\vec{v} \times \vec{w}$.

iii. $\vec{u}$ and $\vec{v}$ are orthogonal.

iv. $\vec{u}$ and $\vec{v}$ are unit vectors.

v. $\dfrac{\vec{u}}{|\vec{v}|}$ is a unit vector.

11. Two lines are defined below using their symmetric equations.

$$x - 1 = 2(y + 1) = 3(z - 2)$$

and

$$x - 3 = 2(y - 1) = 3(z + 1).$$

(a) Show that these lines are parallel.

(b) Find an equation for the plane that contains these lines.

12. Consider the surface defined by the equation,

$$z = \cos(xy).$$

(a) Calculate the equations of the level curves of this surface.

(b) Sketch some of the level curves in the xy plane.

(c) Use the level curves to sketch a picture of the surface in three dimensions.

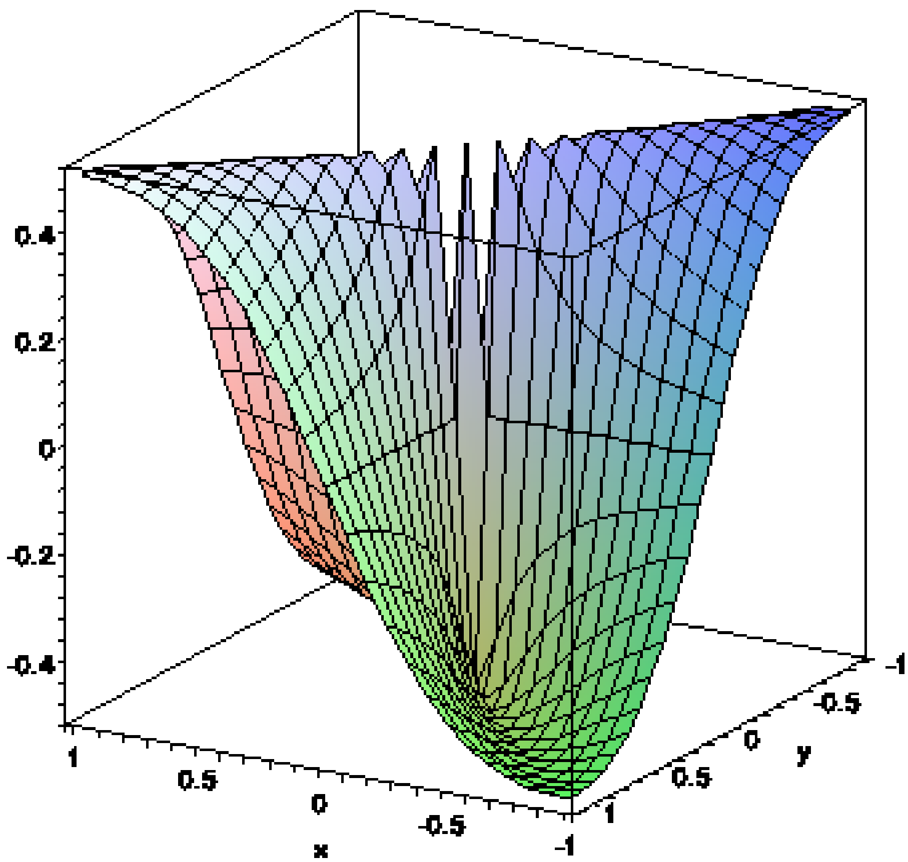
13. Find and classify all of the critical points (local minimums, local maximums, saddle points, etc.) of the surface defined by the equation:

$$f(x, y) = x^2 + xy + y^2 - 6x + 2.$$

14. In this problem, $g(x, y)$ will refer to the function defined by the formula:

$$g(x,y) = \frac{xy}{x^2 + y^2}.$$

(a) The graph $z = g(x, y)$ is shown below. Based on the appearance of this graph, do you think that $g(x, y)$ has a limit as $(x, y) \to (0, 0)$?



(b) Either prove that $g(x, y)$ has a limit as $(x, y) \to (0, 0)$ or show that the limit does not exist. ("Prove" here means use the ε–δ definition.)

15. Describe the level surfaces of the following functions. In each case, prove that the function has a limit as $(x, y) \to (0, 0)$ or show that the limit does not exist. ("Prove" here means use the ε–δ definition.)

(a) $f(x, y) = xy$

(b) $f(x,y) = \dfrac{x + y}{x - y}$

(c) $f(x,y) = e^{1/\left(x^2 + y^2\right)}$

16. Find the equation of the tangent plane to the given surface at the given point.

(a) $z = y^2 - x^2$. (-4, 5, 9)

(b) $z = \ln(2x + y)$. (-1, 3, 0)

(c) $f(x,y) = x^2 + 3y^2 - 2xy$, (2, 2, 8).

17. The temperature, T, in a metal ball is inversely proportional to the distance from the center of the ball (which we will assume is located at the origin). The temperature at the point (1, 2, 2) is 120 degrees centigrade.

(a) Find the rate of change in T at (1, 2, 2) in the direction towards the point (2, 1, 3).

(b) Show that at any point in the ball, the direction of greatest increase in temperature is a vector pointing directly towards the origin.

(c) Find a formula for the greatest possible rate of change of temperature at the point (x, y, z) (where $(x, y, z) \neq (0, 0, 0)$).

18. In this problem the function $f(x, y, z)$ will always refer to the function defined by the formula:

$$f(x,y,z) = 10e^{-0.01\left(3x^2 - y^2 - 2z^2\right)}.$$

This function gives the temperature of a snake (on a plane) in degrees Celsius (ºC). The coordinates of the snake's physical location (x, y, z) are all measured in meters.

(a) The snake is located at the point (1, 1, 1) and plans to slither in the direction given by the vector $\vec{v} = \langle 1,2,3 \rangle$. What is the rate of change of temperature that the snake will experience? Give appropriate units with your answer.

(b) Snakes are cold blooded and enjoy warmth. Consider the snake sitting at the point (1, 1, 1). In what direction should the snake slither to maximize the rate of change of temperature?

(c) What rate of change of temperature will the snake experience if it starts at the point (1, 1, 1) and slithers in the direction you calculated in (b)? Give appropriate units with your answer.

19. In this problem, $z = f(x, y)$ is the function of x and y defined by the following formula:

$$z = f(x,y) = e^{-x+y^2}.$$

(a) Suppose that x and y are both functions of t. All that you can assume about the functions x and y is listed below.

- $x(2) = 1$
- $y(2) = -1$
- $x'(2) = -3$
- $y'(2) = \frac{1}{2}$

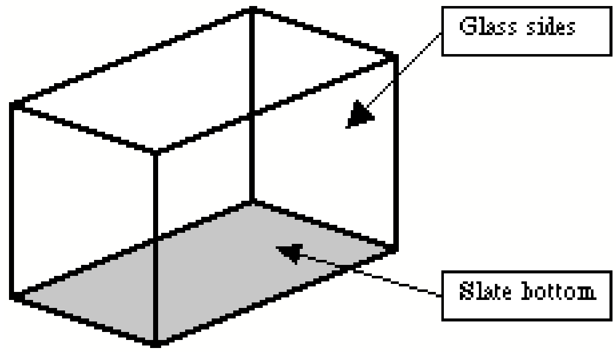
Calculate $z'(2)$.

(b) Suppose instead that x and y are both functions of t and s, i.e. $x = x(t, s)$ and $y = y(t, s)$. All that you can assume about the functions x and y is listed below.

- $x(2,0) = 1$
- $x_t(2,0) = -3$
- $x_s(2,0) = -\frac{1}{2}$
- $y(2,0) = -1$
- $y_t(2,0) = \sqrt{2}$
- $y_s(2,0) = 1$

Calculate $z_s(2,0)$. Show your work!

20. The mantis shrimp (*Squilla empusa*) is a small shrimp-like crustacean with very powerful front claws. These shrimp are sometimes called "thumb splitters" because they can hit so hard with their claws that they sometimes split peoples' thumbs open.



A public aquarium is planning to exhibit mantis shrimp. They will need to build a special tank with a slate bottom and glass sides. The reason for this is that sometimes the shrimp pound on the bottom of the tank with their claws and can break a glass bottom. The shrimp don't jump, so the tank doesn't need a lid.

The tank must have a volume of 1,000,000 cm^3 of water. Glass costs 10 cents per square centimeter, and slate costs 50 cents per square centimeter. Find the dimensions of the least expensive tank that will hold 1,000,000 cm^3 of water.

21. Find the coordinates of the point (or points) on the surface,

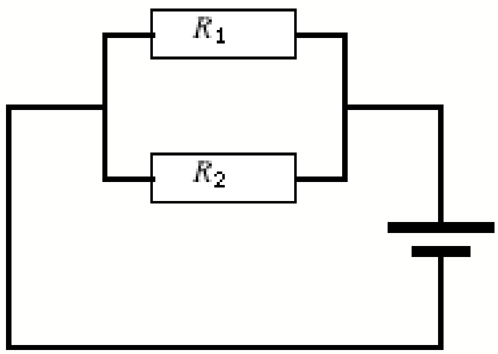
$$xy^2z^3 = 2$$

that are closest to the origin (0, 0, 0).

22. When an electrical current I enters two resistors with resistances R_1 and R_2, that are connected in parallel (see below), it splits into two currents I_1 and I_2 (with $I = I_1 + I_2$) so that the total electrical power,

$$P = R_1 I_1^2 + R_2 I_2^2$$

is minimized.



(a) Find formulas for I_1 and I_2 in terms of I and the two resistances, R_1 and R_2.

(b) Show that the two resistances are equivalent to a single resistance, R, where:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

23. A snake (on a plane) has found a warm metal plate to slither around on. The temperature of the plate (given in °C) at a point (x and y are both measured in meters) is given by the function:

$$T(x,y) = 4x^2 - 4xy + y^2.$$

The snake slithers around in a path that looks exactly like a circle of radius 5 meters centered on the origin. What are the highest and lowest temperatures encountered by the snake as it slithers around this circular path?

24. In this problem the function $f(x, y)$ will always refer to the function defined by the formula:

$$f(x,y) = xy - x - y + 3.$$

(a) Find the x and y coordinates of any critical points of $f(x, y)$.

(b) Classify the critical points that you found in Part (a) as local maximums, local minimums or saddle points.

(c) Find the global maximum and global minimum of $f(x, y)$ on the triangular region of the first quadrant with vertices located at the points (0, 0), (2, 0) and (0, 4).

25. Each of the following integrals is very difficult to work out. Reverse the order of integration to make the integral easier, and then evaluate.

(a) $\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy$

(b) $\int_0^3 \int_{y^2}^9 y\cos\left(x^2\right)dxdy$

(c) $\int_0^1 \int_{\arcsin(y)}^{\pi/2} \cos(x)\sqrt{1+\cos^2(x)}\,dxdy$

26. Find the volumes of the regions described below.

(a) Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$.

(b) Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1) and (1, 2).

(c) Bounded by the cylinder $x^2 + y^2 = 9$ and the planes $x = 0$, $y = 0$, $x + 2y = 2$ in the first octant.

27. Use polar coordinates to evaluate the integrals given below.

(a) $\iint_R xy\,dA$ where R is the region in the first quadrant between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

(b) $\iint_D e^{-x^2-y^2}\,dA$ where D is the region bounded by the semi-circle $x = \sqrt{4 - y^2}$ and the y-axis.

(c) $\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx$.

28. Find the coordinates of the center of mass of a uniform sheet of material in the shape of an isosceles right triangle. The two equal sides both have length a, and the density of the material is proportional to the square of the distance from the vertex opposite the hypoteneuse.

29. The average value of a function $f(x, y, z)$ over a solid region E is defined to be:

$$average = \frac{1}{V(E)} \iiint_E f(x,y,z)\,dV,$$

where $V(E)$ is the volume of the region E. Find the average value of $f(x, y, z) = xyz$ over the cube with side length L with one vertex at the origin and sides parallel to the coordinate axes.

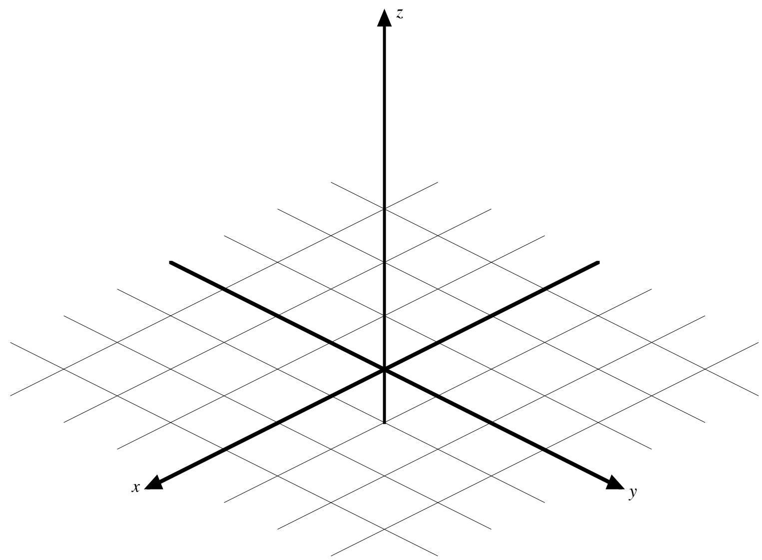
30. **(a)** Calculate the value of the triple integral: $\iiint\limits_B e^{-\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\, dV$, where B is the solid sphere of radius $a > 0$ centered at the origin. Your final answer should contain the letter a.

(b) Consider the vector field $\vec{F}(x,y,z)$ defined by the formula:

$$\vec{F}(x,y,z) = \left\langle -3x\sqrt{x^2+y^2+z^2}\cdot e^{-\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}, -3y\sqrt{x^2+y^2+z^2}\cdot e^{-\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}, -3z\sqrt{x^2+y^2+z^2}\cdot e^{-\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \right\rangle$$

Calculate value of the line integral: $\int\limits_C F \bullet d\vec{r}$, where C is the line segment joining the points (2, 0, 0) and (0, 1, 1).

31. **(a)** The surface S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$. Use the axes provided below to make an accurate sketch of the surface S.



(b) Set up a triple integral in x, y, z coordinates that will give the volume enclosed by the surface S, the cylinder $x^2 + y^2 = 4$ and the plane $z = 0$.

(c) Convert your integral from Part (b) to cylindrical coordinates and use this to calculate the volume enclosed by the surface S, the cylinder $x^2 + y^2 = 4$ and the plane $z = 0$.

32. The diagrams given below show four vector fields. Match each picture of a vector field with one of the formulas given below. You should have one unmatched formula at the end of the problem.

DIAGRAM "A"

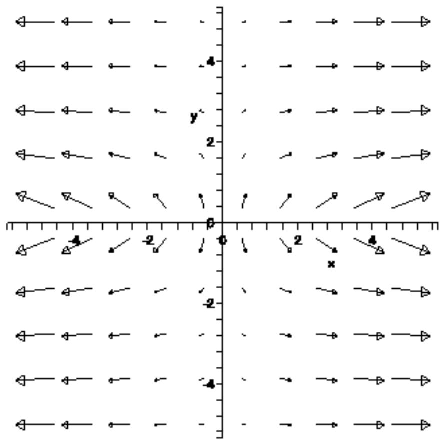


DIAGRAM "B"

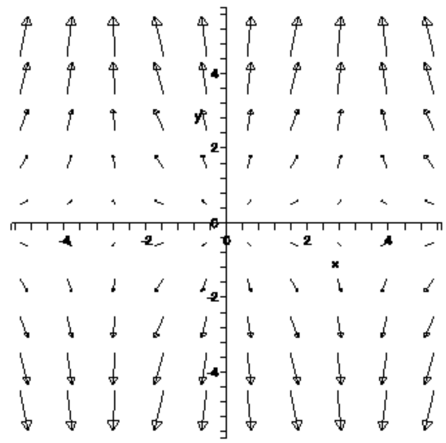


DIAGRAM "C"

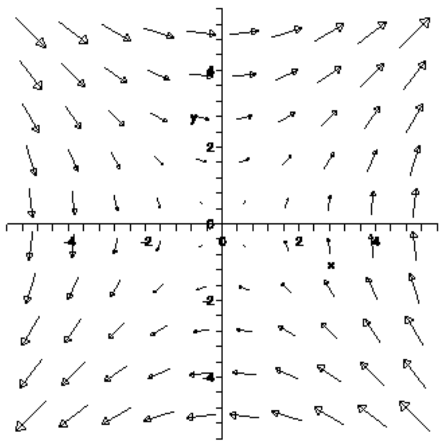
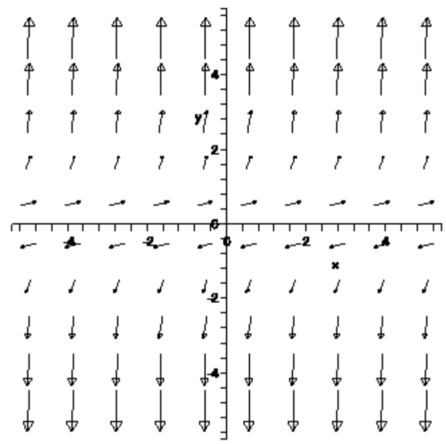


DIAGRAM "D"



(a) $\vec{F}(x,y) = \left\langle x, \frac{1}{y} \right\rangle$ **MATCHING PICTURE:** ____________________

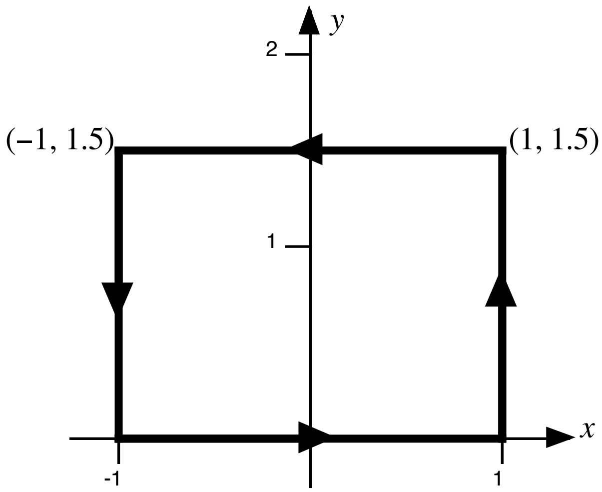
(b) $\vec{F}(x,y) = \left\langle \frac{1}{y}, y \right\rangle$ **MATCHING PICTURE:** ____________________

(c) $\vec{F}(x,y) = \left\langle \sin(x), y \right\rangle$ **MATCHING PICTURE:** ____________________

(d) $\vec{F}(x,y) = \left\langle \frac{1}{y}, x \right\rangle$ **MATCHING PICTURE:** ____________________

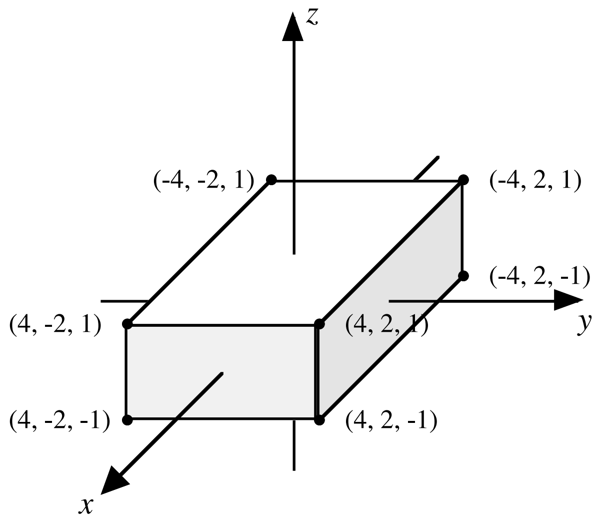
(e) $\vec{F}(x,y) = \left\langle y, x \right\rangle$ **MATCHING PICTURE:** ____________________

33. **(a)** Evaluate the line integral: $\int_C y^2 dx + 3xy \cdot dy$ **directly** (i.e. not using Green's Theorem) where C is the curve shown in the diagram given below.



(b) Evaluate the line integral: $\int_C y^2 dx + 3xy \cdot dy$ using Green's Theorem.

34. Evaluate the surface integral $\iint_S \langle 3x, 4y, 5z \rangle \bullet d\vec{S}$, where S is the surface of the rectangular prism shown in the diagram given below with positive orientation. (The vertex of the prism obscured in the diagram is $(-4, -2, -1)$.)



35. **(a)** Evaluate the surface integral $\iint_S f(x,y,z)dS$, where $f(x,y)=1+xy$ and S is the part of the paraboloid $z=x^2+y^2$ that lies inside the cylinder $x^2+y^2=4$.

NOTE: The following integral formula may be helpful:

$$\int \sqrt{r^2+4r^4}\,dr = \frac{\left(1+4r^2\right)\sqrt{r^2+4r^4}}{12r}+C.$$

(b) Evaluate the surface integral: $\iint_S \langle xz,-yz,-5\rangle \bullet d\vec{S}$, where S is the hemispherical surface $z=\sqrt{4-x^2-y^2}$.

NOTE: The following integral formula may be helpful:

$$\int \sin^2(x)dx = \tfrac{1}{2}x-\tfrac{1}{4}\sin(2x)+C \qquad \text{and} \qquad \int \cos^2(x)dx = \tfrac{1}{2}x+\tfrac{1}{4}\sin(2x)+C.$$