SOLUTIONS

Math 259 **Winter 2009**

Handout 5: In-class Review for Exam 3

The topics covered by Exam 3 in the course include the following:

- Lagrange Multipliers.
- Lagrange Multipliers with two constraints.
- Word problems that can be solved using Lagrange Multipliers.
- Double Riemann sums.
- The Midpoint Rule for double Riemann sums (no error estimates).
- Setting up double integrals.
- Changing the order of integration in a double integral.
- Evaluating double integrals.
- Applications of double integrals to chemistry and physics (e.g. volume, mass, center of mass).
- Polar coordinates.
- Converting double integrals from Cartesian to polar coordinates.
- Setting up double integrals in polar coordinates.
- Evaluating double integrals in polar coordinates.
- Setting up triple integrals.
- Changing the order of integration for a triple integral.
- Evaluating triple integrals.
- Applications of triple integrals (e.g. calculating volume and mass).
- Setting up and evaluating triple integrals in cylindrical/polar coordinates.
- Setting up and evaluating triple integrals in spherical coordinates.
- Drawing two-dimensional vector fields.
- Associating formulas for two-dimensional vector fields with visual representations.
- Determining whether a vector field is conservative or not.
- Finding potential functions for conservative vector fields.
- Setting up parametric equations for curves in two dimensions.
- Setting up and evaluating (directly) line integrals of functions in two dimensions.
- Setting up and evaluating (directly) line integrals of vector fields in two and three dimensions.
- Calculating line integrals of conservative vector fields using the Fundamental Theorem.
- Calculating line integrals in two dimensions using Green's Theorem.

1. Find the volume of the solid bounded by the surface:

$$z = x\sqrt{x^2 + y}$$

and the planes: $x = 0, x = 1, y = 0, y = 1$, and $z = 0$.

$$\text{Volume} = \int_0^1 \int_0^1 x \cdot \sqrt{x^2 + y}\; dx\, dy$$

$$= \int_0^1 \left[\frac{1}{2} \cdot \frac{2}{3} (x^2 + y)^{3/2} \right]_0^1 dy$$

$$= \int_0^1 \frac{1}{3} \left((1+y)^{3/2} - y^{3/2} \right) dy$$

$$= \frac{2}{15} \left[(1+y)^{5/2} - y^{5/2} \right]_0^1$$

$$\approx 0.4875805666$$

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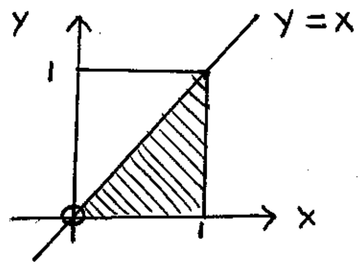
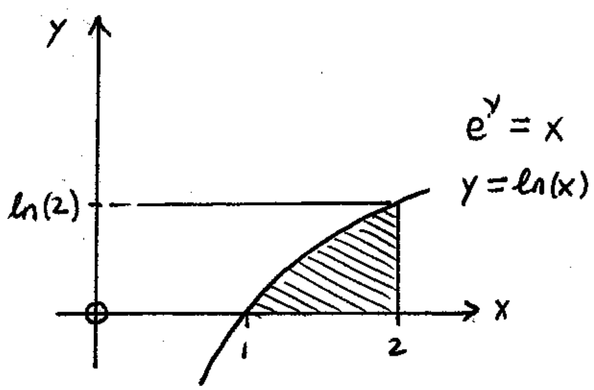
2. The following integrals are given with the function $f(x, y)$ unspecified. Suppose that the function $f(x, y)$ was specified and the formula for $f(x, y)$ is one where the integral would be a lot easier to evaluate if the dx and dy were swapped. Write down what the new integrals would be (with the order of dx and dy swapped).

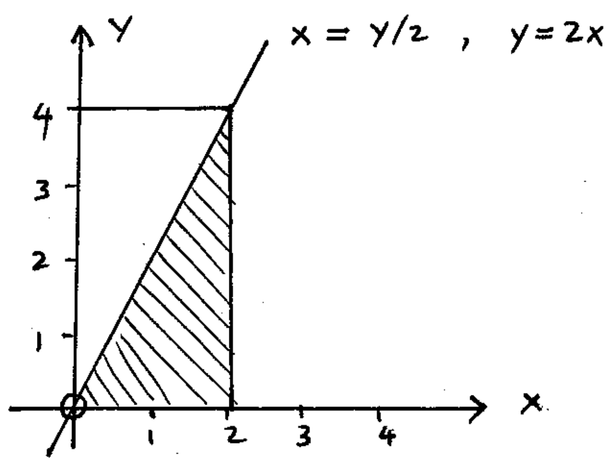
(a) $\int_0^1\int_0^x f(x,y)\,dy\,dx \quad = \quad \int_0^1 \int_y^1 f(x,y)\,dx\,dy$



(b) $\int_1^2\int_0^{\ln(x)} f(x,y)\,dy\,dx \quad = \quad \int_0^{\ln(2)} \int_{e^y}^2 f(x,y)\,dx\,dy$



(c) $\int_0^4\int_{y/2}^2 f(x,y)\,dx\,dy \quad = \quad \int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$



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3. Evaluate the line integral,

$$\int_C y^2\,dx + x\,dy$$

(a) When C is the line segment from (-5, -3) to (0, 2).

$x(t) = (1-t)(-5) + t(0) = -5 + 5t$

$y(t) = (1-t)(-3) + t(2) = -3 + 5t$ $\quad 0 \le t \le 1.$

$$\int_C y^2\,dx = \int_0^1 (-3+5t)^2 \cdot 5 \cdot dt = \left[\tfrac{1}{3}(-3+5t)^3\right]_0^1 = \frac{8}{3} + \frac{27}{3} = \frac{35}{3}$$

$$\int_C x\,dy = \int_0^1 (-5+5t)\cdot 5 \cdot dt = \left[-25t + \frac{25}{2}t^2\right]_0^1 = \frac{-25}{2}$$

$$\int_C y^2\,dx + x\,dy = \frac{35}{3} - \frac{25}{2} = \frac{-5}{6}.$$

(b) When C is the part of the parabola from $x = 4 - y^2$ from (-5, -3) to (0, 2).

$x(t) = 4 - t^2$

$y(t) = t$ $\quad -3 \le t \le 2.$

$$\int_C y^2\,dx = \int_{-3}^{2} t^2\cdot(-2t)\,dt = \left[\frac{-1}{2}t^4\right]_{-3}^{2} = \frac{65}{2}$$

$$\int_C x\,dy = \int_{-3}^{2}(4-t^2)(1)\,dt = \left[4t - \frac{1}{3}t^3\right]_{-3}^{2} = \frac{25}{3}$$

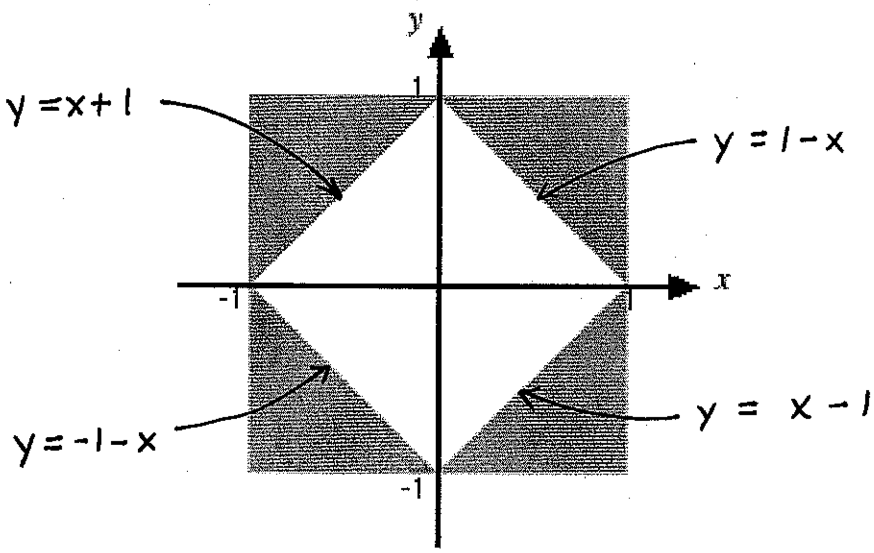
$$\int_C y^2\,dx + x\,dy = \frac{65}{2} + \frac{25}{3} = \frac{245}{6}$$

(c) How could you tell that the two integrals would yield different values without evaluating the integrals?

The vector field $\vec{F}(x,y) = \langle y^2, x\rangle$ is not conservative, so the line integral is not independent of path.

4. The region of integration is the shaded region D shown in the diagram below. Find the value of the integral,

$$\iint_D x^2\,dxdy\,.$$



$$\iint_D x^2\,dx\,dy = \int_0^1 \int_{-1}^{y-1} x^2\,dx\,dy + \int_0^1 \int_{1-y}^{1} x^2\,dx\,dy$$

$$+ \int_{-1}^{0} \int_{-1}^{-1-y} x^2\,dx\,dy + \int_{-1}^{0} \int_{y+1}^{1} x^2\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{3}(y-1)^3 + \tfrac{1}{3}\,dy + \int_0^1 \tfrac{1}{3} - \tfrac{1}{3}(1-y)^3\,dy$$

$$+ \int_{-1}^{0} \tfrac{-1}{3}(1+y)^3 + \tfrac{1}{3}\,dy + \int_{-1}^{0} \tfrac{1}{3} - \tfrac{1}{3}(y+1)^3\,dy$$

$$= \left[\tfrac{1}{12}(y-1)^4 + \tfrac{1}{3}y\right]_0^1 + \left[\tfrac{1}{3}y + \tfrac{1}{12}(1-y)^4\right]_0^1$$

$$+ \left[\tfrac{1}{3}y - \tfrac{1}{12}(1+y)^4\right]_{-1}^{0} + \left[\tfrac{1}{3}y - \tfrac{1}{12}(y+1)^4\right]_{-1}^{0}$$

$$= \frac{1}{3} - \frac{1}{12} + \frac{1}{3} - \frac{1}{12} - \frac{1}{12} + \frac{1}{3} - \frac{1}{12} + \frac{1}{3}$$

$$= 2/3.$$

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5. Decide whether or not the following vector fields are conservative. If any are conservative, find a function f so that $\nabla f = \mathbf{F}$.

(a) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ \partial/\partial x & \partial/\partial y \\ yz & xz \end{array}$$

$$= \langle x - x, y - y, z - z \rangle$$

$$= \vec{0}$$

$\vec{F}(x,y,z)$ is conservative. $f(x,y,z) = xyz$.

(b) $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & (x^2+2yz) & y^2 \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ \partial/\partial x & \partial/\partial y \\ 2xy & (x^2+2yz) \end{array}$$

$$= \langle 2y - 2y, 0, 2x - 2x \rangle$$

$$= \vec{0}.$$

$\vec{F}(x,y,z)$ is conservative. $f(x,y,z) = x^2 y + y^2 z$.

(c) $\mathbf{F}(x, y, z) = e^x\mathbf{i} + e^y\mathbf{j} + e^z\mathbf{k}$.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \left|\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & e^y & e^z \end{array}\right| \begin{array}{cc} \vec{i} & \vec{j} \\ \partial/\partial x & \partial/\partial y \\ e^x & e^y \end{array}$$

$$= \langle 0, 0, 0 \rangle.$$

$\vec{F}(x,y,z)$ is conservative. $f(x,y,z) = e^x + e^y + e^z$.

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6. Evaluate the line integral, where C is the given curve.

(a) $\int_C y\,ds$, where C: $x = t^2$, $y = t$, $0 \le t \le 2$.

$$\int_C y\,ds = \int_0^2 t \cdot \sqrt{4t^2+1}\,dt$$

$$= \left[\frac{1}{8} \cdot \frac{2}{3} \left(4t^2+1\right)^{3/2} \right]_0^2$$

$$= 5.75773297$$

(b) $\int_C xy^4\,ds$, where C is the right hand side of the circle $x^2 + y^2 = 16$.

$$x(t) = 4\cos(t) \qquad y(t) = 4 \cdot \sin(t) \qquad -\pi/2 \le t \le \pi/2.$$

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} 4^5 \cdot \cos(t) \cdot \sin^4(t) \cdot \sqrt{16}\,dt$$

$$= \left[\frac{4^6}{5} \sin^5(t) \right]_{-\pi/2}^{\pi/2}$$

$$= (2)(4^6)/5.$$

(c) $\int_C (xy + \ln(x))\,dy$, where C is the arc of the parabola $y = x^2$ from (1, 1) to (3, 9).

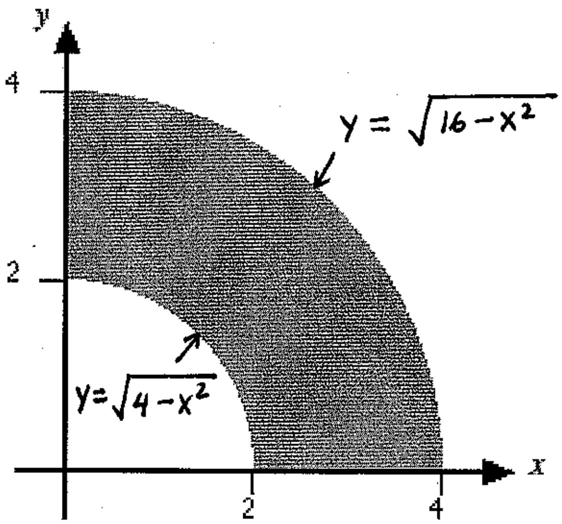
$$x(t) = t \qquad y(t) = t^2 \qquad 1 \le t \le 3.$$

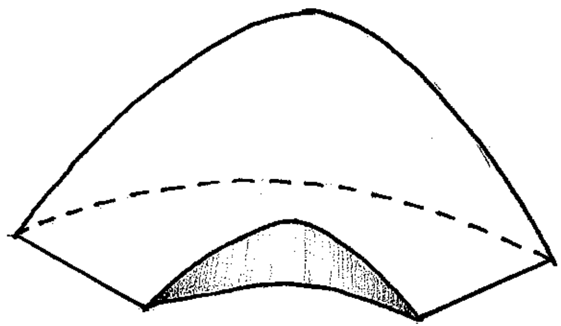
$$\int_C (xy + \ln(x))\,dy = \int_1^3 (t^3 + \ln(t)) \cdot 2t \cdot dt$$

$$= \left[\frac{2}{5} t^5 + \frac{2t^2}{4}\left(2\ln(t) - 1\right) \right]_1^3$$

$$= 102.6875106$$

7. **(a)** Sketch the solid that has a height given by: $z = xy$, and a base given by the shaded region in the diagram below.





(b) Set up in integral (in terms of x, y, dx, dy) that gives the volume of the solid.

$$\text{Volume} = \int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{16-x^2}} xy \, dy\, dx + \int_2^4 \int_0^{\sqrt{16-x^2}} xy \, dy\, dx$$

(c) Evaluate the integral to find the volume.

$$\text{Volume} = \int_0^{\pi/2} \int_2^4 r^2 \cos(\theta) \sin(\theta) \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{4} r^4 \cos(\theta) \sin(\theta) \right]_2^4 d\theta$$

$$= \left[30 \cdot \sin^2(\theta) \right]_0^{\pi/2}$$

$$= 30.$$

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8. Find the global maximum and global minimum values of the function:

$$f(x,y,z) = 3x - y - 3z$$

subject to the constraints:

$$x + y - z = 0 \qquad \text{and} \qquad x^2 + 2z^2 = 1.$$

$g(x,y,z) = x + y - z \qquad h(x,y,z) = x^2 - 2z^2 - 1.$

$\nabla f = \langle 3, -1, -3 \rangle$

$\nabla g = \langle 1, 1, -1 \rangle$

$\nabla h = \langle 2x, 0, -4z \rangle.$

$\underline{\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h:}$

$$\begin{aligned} 3 &= \lambda_1 + 2\lambda_2 x \\ -1 &= \lambda_1 \\ -3 &= -\lambda_1 - 4\lambda_2 z \end{aligned}$$

Since $\lambda_1 = -1$ we can rewrite this as:

$$\begin{aligned} \lambda_2 x &= 2 \\ \lambda_2 z &= -1 \end{aligned}$$

so that $x = -2z$. Substituting this into $x^2 + 2z^2 = 1$ gives:

$$6z^2 = 1 \qquad z = \pm 1/\sqrt{6}.$$

$$x = 2/\sqrt{6}.$$

We can then solve for y using $y = z - x$.

x	y	z	$f(x,y,z)$	Comments
$-2/\sqrt{6}$	$3/\sqrt{6}$	$1/\sqrt{6}$	$-12/\sqrt{6}$	Global min
$+2/\sqrt{6}$	$-3/\sqrt{6}$	$-1/\sqrt{6}$	$12/\sqrt{6}$	Global max

9. Three alleles (alternative version of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO) and AB. The Hardy-Weinberg law states that the proportion of individuals in a population who carry two different alleles is:

$$P = 2pq + 2pr + 2rq$$

where p, q, and r are the proportions of A, B, and O in the population. Use the fact that $p + q + r = 1$ to show that P is at most two thirds.

We will maximize $f(p,q,r) = 2pq + 2pr + 2rq$ subject to the constraint $g(p,q,r) = p + q + r - 1$.

$\nabla f = \langle 2q + 2r, 2p + 2r, 2p + 2q \rangle$

$\nabla g = \langle 1, 1, 1 \rangle.$

$\underline{\nabla f = \lambda \cdot \nabla g:}$

$$2q + 2r = \lambda$$
$$2p + 2r = \lambda$$
$$2p + 2q = \lambda$$

The solution of this system is: $p = q = r$. Substituting into the constraint equation gives:

$$p = q = r = \frac{1}{3}.$$

The global maximum of $f(p,q,r)$ is then:

$$f\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = 2\left(\tfrac{1}{3}\right)^2 + 2\left(\tfrac{1}{3}\right)^2 + 2\left(\tfrac{1}{3}\right)^2$$
$$= \frac{2}{3}.$$

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10. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

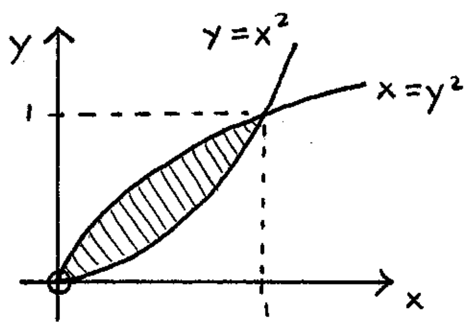
(a) $\int_C e^y dx + 2xe^y dy$, where C is the square with sides $x = 0$, $x = 1$, $y = 0$ and $y = 1$.

$$P(x,y) = e^y \qquad \frac{\partial P}{\partial y} = e^y \qquad Q(x,y) = 2xe^y \qquad \frac{\partial Q}{\partial x} = 2e^y$$

$$\int_C e^y dx + 2xe^y dy = \int_0^1 \int_0^1 e^y \, dy \, dx$$

$$= e^1 - 1.$$

(b) $\int_C \left(y + e^{\sqrt{x}}\right)dx + \left(2x + \cos\left(y^2\right)\right)dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



$$P(x,y) = y + e^{\sqrt{x}} \qquad Q(x,y) = 2x + \cos(y^2)$$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 2$$

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

(c) $\int_C y^3 dx + x^3 dy$, where C is the circle $x^2 + y^2 = 4$.

$$P(x,y) = y^3 \qquad \frac{\partial P}{\partial y} = 3y^2 \qquad Q(x,y) = x^3 \qquad \frac{\partial Q}{\partial x} = 3x^2$$

$$\int_C y^3 dx + x^3 dy = \int_0^{2\pi} \int_0^2 3r^2(\cos^2(\theta) - \sin^2(\theta))\, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{4} r^4 \cos(2\theta)\right]_0^2 d\theta$$

$$= \left[6 \cdot \sin(2\theta)\right]_0^{2\pi} = 0.$$