SOLUTIONS

Handout 4: In-Class Review for Exam 2

The topics covered by Exam 2 in the course include the following:

- Sketching surfaces in 3D using contour plots.
- Interpreting contour plots.
- Classifying quadric surfaces.
- Recognizing planes, quadric surfaces and cylinders from their equations.
- Creating and using vector functions in 3D.
- Velocities and tangent vectors for vector functions (including unit tangent vectors).
- Showing that limits in 2D do not exist using a variety of strategies (e.g. $y = mx$, contour plots or tables).
- Evaluating and using functions with several input variables.
- Proving that limits do exist using the ε-δ definition.
- Calculating values for limits/showing limits exist using the Squeezing Theorem.
- Calculating and interpreting partial derivatives.
- Finding equations for tangent planes.
- Using the tangent plane to calculate a linear approximation.
- Calculating total differentials for functions.
- Using total differentials to estimate changes and errors.
- Using the Chain Rule for functions of several variables.
- Calculating directional derivatives for functions.
- Calculating gradient vectors.
- Finding the direction of maximum rate of change (and magnitude of the maximum rate of change).
- Interpreting the practical meaning of a directional derivative.
- Finding and classifying (local maximum, local minimum, saddle point) the critical points of a function of several variables using partial derivatives and the Jacobian determinant.
- Finding the global maximum and global minimum of a continuous function over a region in the xy-plane.

1. In this problem, $f(x, y)$ will always refer to the function defined below.

$$f(x,y) = x^2 + 3y^2 - 2xy.$$

(a) Calculate $f_x(x,y)$.

$$f_x(x,y) = 2x - 2y$$

(b) Calculate $f_y(x,y)$.

$$f_y(x,y) = 6y - 2x$$

Continued on the next page.

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(c) Find an equation for the tangent plane to the surface $z = f(x, y)$ at the point $(2, 2, 8)$.

$$f_x(2,2) = 0 \qquad f_y(2,2) = 8$$

Equation of tangent plane:

$$8(y-2) - (z-8) = 0.$$

(d) Use your answer from Part (c) to estimate the value of $f(1.9, 2.1)$. The answer I am looking for here is **not** 8.86.

Equation of tangent plane:

$$z = 8 + 8(y-2)$$

$$f(1.9, 2.1) \approx 8 + 8(2.1 - 2) = 8.8.$$

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2. In this problem, $z = f(x, y)$ is the function of x and y defined by the following formula:

$$z = f(x,y) = e^{-x+y^2}.$$

(a) Suppose that x and y are both functions of t. All that you can assume about the functions x and y is listed below.

- $x(2) = 1$
- $x'(2) = -3$
- $y(2) = -1$
- $y'(2) = \frac{1}{2}$

Calculate $z'(2)$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$z'(2) = (-1) \cdot e^{-1+(-1)^2} \cdot (-3) + (2)(-1) e^{-1+(-1)^2} \cdot \left(\frac{1}{2}\right)$$

$$= 3 - 1$$

$$= 2$$

(b) Suppose instead that x and y are both functions of t and s, i.e. $x = x(t, s)$ and $y = y(t, s)$. All that you can assume about the functions x and y is listed below.

- $x(2,0) = 1$
- $x_t(2,0) = -3$
- $x_s(2,0) = -\frac{1}{2}$
- $y(2,0) = -1$
- $y_t(2,0) = \sqrt{2}$
- $y_s(2,0) = 1$

Calculate $z_s(2,0)$. Show your work!

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$z_s(2,0) = (-1) \cdot e^{-1+(-1)^2} \cdot \left(\frac{-1}{2}\right) + (2)(-1) e^{-1+(-1)^2} \cdot (1)$$

$$= \frac{1}{2} - 2$$

$$= -3/2.$$

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3. In this problem you will be working with the curve defined by the vector function:

$$\vec{r}(t) = \langle \cos(t)\cdot \sin(t), \sin(t), t \rangle.$$

(a) Find a formula for the unit tangent vector to this curve, $\vec{T}(t)$.

$$\vec{r}\,'(t) = \langle -\sin^2(t) + \cos^2(t)\ ,\ \cos(t)\ ,\ 1 \rangle$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{(\cos^2(t) - \sin^2(t))^2 + \cos^2(t) + 1}} \langle \cos^2(t) - \sin^2(t), \cos(t), 1 \rangle$$

(b) Calculate the tangent vector to the curve at the point where $t = \frac{\pi}{2}$.

$$\vec{r}\,'(\tfrac{\pi}{2}) = \langle -1\ ,\ 0\ ,\ 1 \rangle$$

(c) Write down an equation for the line that is tangent to this curve at the point where $t = \frac{\pi}{2}$.

$$\vec{r}(\pi/2) = \langle 0, 1, \pi/2 \rangle.$$

Equation of tangent line:

$$\langle x, y, z \rangle = \langle 0, 1, \pi/2 \rangle + t\cdot \langle -1, 0, 1 \rangle.$$

(d) Set up (but do not evaluate) an integral that will give the length of the curve between the points (0, 0, 0) and (0, 1, $\frac{\pi}{2}$). Your answer should not include any vectors or vector functions.

$$\text{Length} = \int_0^{\pi/2} \sqrt{(\cos^2(t) - \sin^2(t))^2 + \cos^2(t) + 1}\ dt$$

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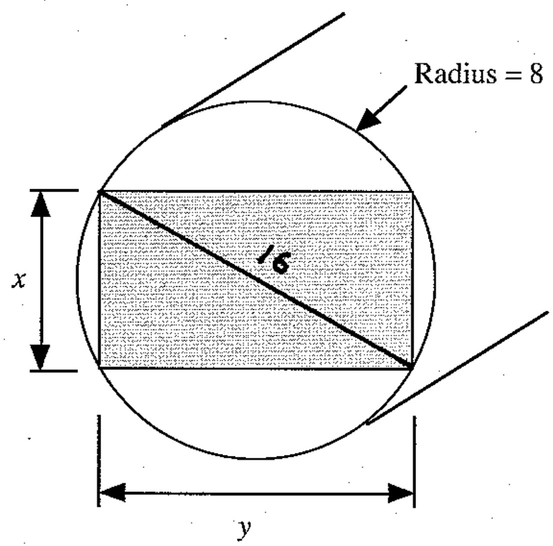
4. A sawmill is planning to cut a log into a beam. The log has a cylindrical cross-section with a radius of eight (8) inches. The beam will have a rectangular cross-section with a height of x and a width of y. The strength, S, of the finished beam will be given by the formula:

$$S = 5xy^2.$$

What values of x and y should be used for the beam if the people at the sawmill wish to make the strongest possible beam?



By Pythagoras, $x^2 + y^2 = 16^2$ so:

$$S = 5x(16^2 - x^2)$$

$$= 1280x - 5x^3$$

$\frac{dS}{dx} = 0$ for $x = \pm\sqrt{\frac{1280}{15}} \approx 9.2376$ inches.

$\frac{d^2S}{dx^2} = -30x$ so have a local max for $x \approx +9.2376$ inches.

Check endpoints: $x = 0$ gives $S = 0$

$x = 16$ gives $S = 0$

Maximum strength when $x \approx +9.2376$ and $y \approx 13.0639$ inches.

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5. In this problem the function $f(x, y, z)$ will always refer to the function defined by the formula:

$$f(x,y,z) = 37e^{-0.1(x^2+y^2-2z^2)}.$$

This function gives the temperature of a snake (on a plane, not a 2D plane) in degrees Celsius (°C). The coordinates of the snake's physical location (x, y, z) are all measured in meters.

(a) The snake is located at the point (1, 1, 1) and plans to slither in the direction given by the vector $\vec{v} = \langle 1,2,3 \rangle$. What is the rate of change of temperature that the snake will experience? Give appropriate units with your answer.

$$\nabla f = \langle 37e^{-0.1(x^2+y^2-2z^2)}(-0.2)x,\ 37e^{-0.1(x^2+y^2-2z^2)}(-0.2)y,\ 37e^{-0.1(x^2+y^2-2z^2)}(0.4)z \rangle$$

$$\nabla f(1,1,1) = \langle -7.4, -7.4, 14.8 \rangle$$

$$\vec{u} = \frac{1}{\sqrt{1^2+2^2+3^2}} \langle 1, 2, 3 \rangle = \langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle$$

$$D_{\vec{u}} f(1,1,1) = \langle -7.4, -7.4, 14.8 \rangle \cdot \langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle = 5.933\ °C/m$$

(b) Snakes are cold blooded and enjoy warmth. Consider the snake sitting at the point (1, 1, 1). In what direction should the snake slither to maximize the rate of change of temperature?

$$\nabla f(1,1,1) = \langle -7.4, -7.4, 14.8 \rangle.$$

(c) What rate of change of temperature will the snake experience if it starts at the point (1, 1, 1) and slithers in the direction you calculated in (b)? Give appropriate units with your answer.

$$|\nabla f(1,1,1)| = \sqrt{(-7.4)^2 + (-7.4)^2 + (14.8)^2}$$

$$\doteqdot 18.126\ °C/m$$

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6. In this problem the function $f(x, y)$ will always refer to the function defined by the formula:

$$f(x,y) = x^2 + y^2 - x - y + 1.$$

(a) Find the x and y coordinates of any critical points of $f(x, y)$.

$$\frac{\partial f}{\partial x} = 2x - 1$$

$$\frac{\partial f}{\partial y} = 2y - 1$$

The only critical point is $(x,y) = (1/2, 1/2)$.

(b) Classify the critical points that you found in Part (a) as local maximums, local minimums or saddle points.

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$D\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = (2)(2) - (0)^2 = 4 > 0$$

$$f_{xx}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2 > 0$$

$(x,y) = (1/2, 1/2)$ is a local minimum.

(c) Find the global maximum and global minimum of $f(x, y)$ on the disk where $x^2 + y^2 \le 1$.

On boundary of disk $x^2+y^2=1$ so $y^2 = 1-x^2$.

$$f_I(x) = x^2 + 1 - x^2 - x \pm \sqrt{1-x^2} + 1$$

$$f_I'(x) = -1 \pm \frac{x}{\sqrt{1-x^2}} = 0 \quad \text{when } x = \pm \frac{1}{\sqrt{2}}$$

and $y = \pm \frac{1}{\sqrt{2}}$.

x	y	$f(x,y)$
$1/2$	$1/2$	$1/2$
$1/\sqrt{2}$	$1/\sqrt{2}$	$2 - 2/\sqrt{2}$
$-1/\sqrt{2}$	$1/\sqrt{2}$	2
$1/\sqrt{2}$	$-1/\sqrt{2}$	2
$-1/\sqrt{2}$	$-1/\sqrt{2}$	$2 + 2/\sqrt{2}$

The global maximum is $2 + 2/\sqrt{2}$.

The global minimum is $1/2$.

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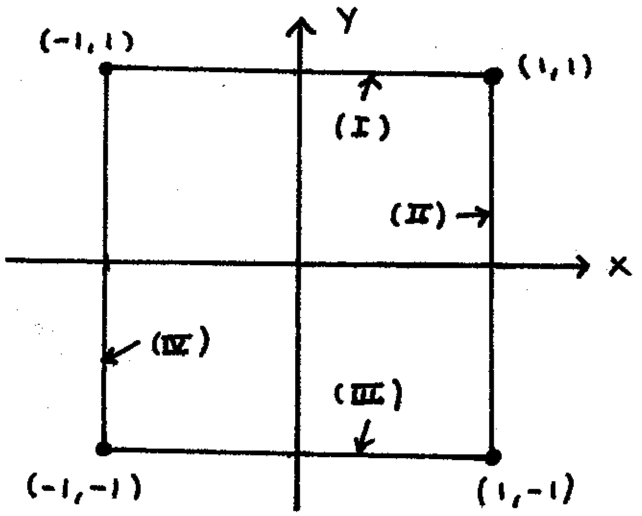
7. Find the global maximum and global minimum of $f(x, y) = x^2 + xy - y^2$ over the region R of the xy-plane that consists of the square with vertices at $(\pm 1, \pm 1)$.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + y = 0 \\ \frac{\partial f}{\partial y} &= x - 2y = 0 \end{aligned}\right\}$$

The only point at which both partial derivatives are zero is $(x, y) = (0, 0)$.

There are no points where the partial derivatives are undefined.



On (I) $y = 1$ so:

$f_I(x) = x^2 + x - 1$

$f_I'(x) = 2x + 1 = 0$ for

$x = -1/2,\ y = 1.$

On (II) $x = 1$ so:

$f_{II}(y) = 1 + y - y^2$

$f_{II}'(y) = 1 - 2y = 0$ for

$y = 1/2,\ x = 1.$

On (III) $y = -1$ so $f_{III}(x) = x^2 - x - 1$ and $f_{II}'(x) = 2x - 1 = 0$ at $x = 1/2$, $y = -1$.

On (IV), $x = -1$ so $f_{IV}(y) = 1 - y - y^2$ and $f_{IV}'(y) = -1 - 2y$ which equals zero at $y = -1/2$, $x = -1$.

x	y	f(x, y)
0	0	0
-1/2	1	-1 1/4
1	1/2	1 1/4
1/2	-1	-1 1/4
-1	-1/2	1 1/4
-1	1	-1
-1	-1	1
1	1	1
1	-1	-1

The global max is $5/4$.

The global min is $-5/4$.

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8. In this problem you will consider the surface defined by the equation:

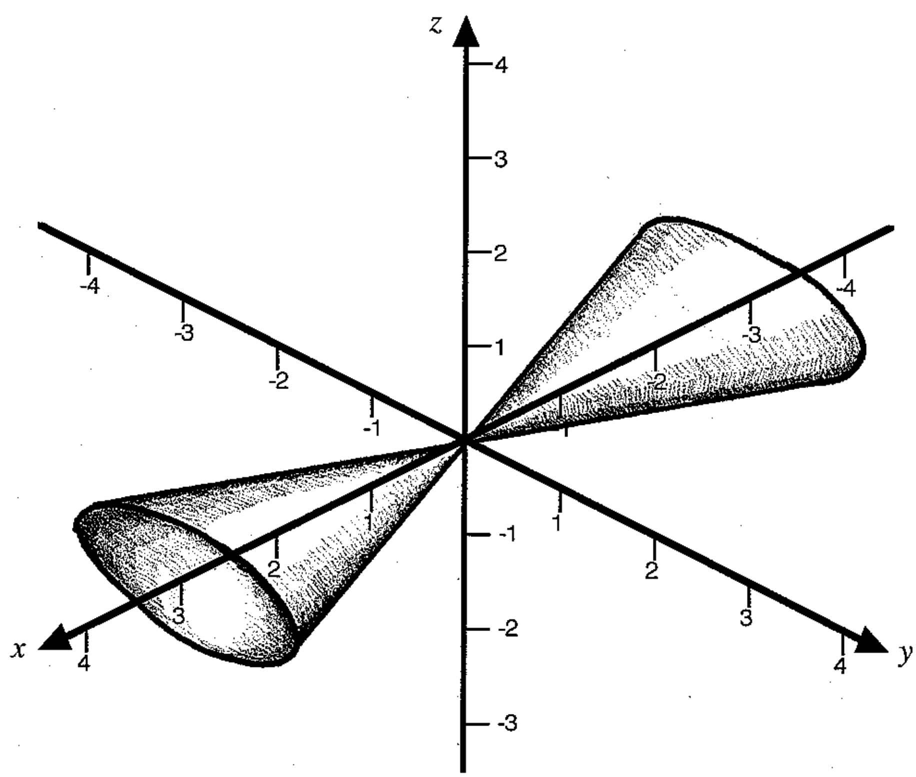
$$x^2 = 4y^2 + 9z^2.$$

(a) Classify the surface (cylinder, cone, elliptic paraboloid, etc.).

This is a cone that opens along the positive and negative x-axes.

The cross-sections of the cone parallel to the yz-plane are ellipses.

(b) Use the axes given below to draw an accurate sketch of the surface. On your sketch, label any interesting points such as vertices and intercepts.



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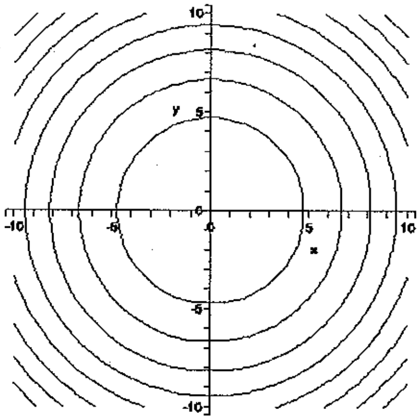
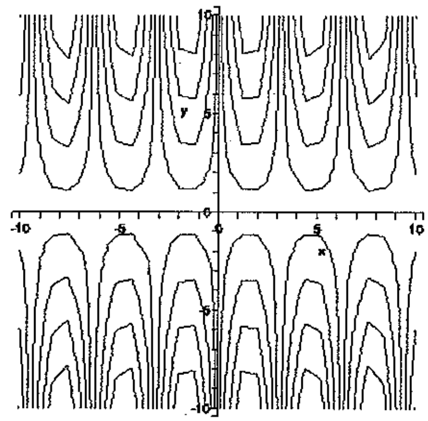
10. Match these plots with the functions given below. You should have one unmatched function left over at the end of the problem.

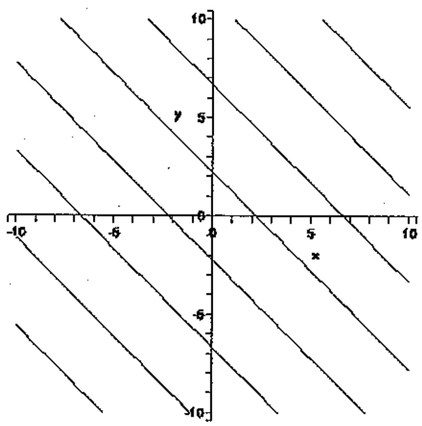
CONTOUR PLOT I



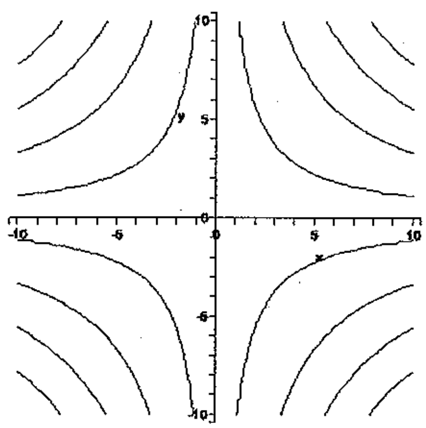
CONTOUR PLOT II



CONTOUR PLOT III



CONTOUR PLOT IV



(a) $f(x,y) = y \cdot \sin(x)$ **CONTOUR PLOT =** II

(b) $f(x,y) = \sin(x) + \sin(y)$ **CONTOUR PLOT =** NONE

(c) $f(x,y) = 4 - x^2 - y^2$ **CONTOUR PLOT =** I

(d) $f(x,y) = x + y$ **CONTOUR PLOT =** III

(e) $f(x,y) = x \cdot y$ **CONTOUR PLOT =** IV