SOLUTIONS

Math 259 **Winter 2009**

Handout 2: In-Class Review for Exam 1

The topics covered by Exam 1 in the course include the following:

- Parametric curves.
- Finding formulas for parametric curves.
- Drawing graphs of curves defined by parametric equations.
- Finding tangent lines to curves defined by parametric equations.
- Finding the area beneath (between the curve and the x-axis) a parametric curve.
- Finding the arc length of a parametric curve.
- Polar coordinates for the xy-plane.
- Identifying regions of the xy-plane described by polar coordinates.
- Converting Cartesian equations to polar equations.
- Converting polar equations to Cartesian equations.
- Sketching curves in the xy-plane defined by polar equations.
- Finding formulas for tangent lines to curves defined by polar equations.
- Finding areas enclosed by polar curves.
- Finding arc lengths of curves defined by polar equations.
- Conic sections in Cartesian and polar coordinates.
- Sketching conic sections defined by polar equations. Identifying eccentricity, directrix, etc. from a polar equation. Classifying conic sections using eccentricity.
- Equations of lines, planes and spheres in 3D.
- Combining vectors. Magnitude of a vector. Unit vectors.
- Applications of vectors in physics.
- Dot product of vectors. Angle between vectors. Orthogonality. Vector projections.
- Cross product of vectors. Geometry of the cross product. Cross product and areas.
- Calculating volumes with the scalar triple product.
- Finding equations for lines and planes in 3D using the cross product.
- Distances from points to lines and planes, and from lines to planes.
- Symmetric equations.

1. For each of the curves defined below:

(i) $r = \dfrac{9}{6 + 2\cdot\cos(\theta)}$ (ii) $r = \dfrac{9}{2\cdot\cos(\theta)}$,

(a) Determine the eccentricity.

(b) Identify the type of curve.

(c) Sketch an accurate graph of the curve using the axes provided.

(i) $r = \dfrac{3/2}{1 + \frac{1}{3}\cos(\theta)}$

(a) eccentricity $= 1/3$

(b) Ellipse.

$r \cdot \cos(\theta) = 9/2$

$x = 9/2$

(a) Not a conic section.

(b) Vertical line.

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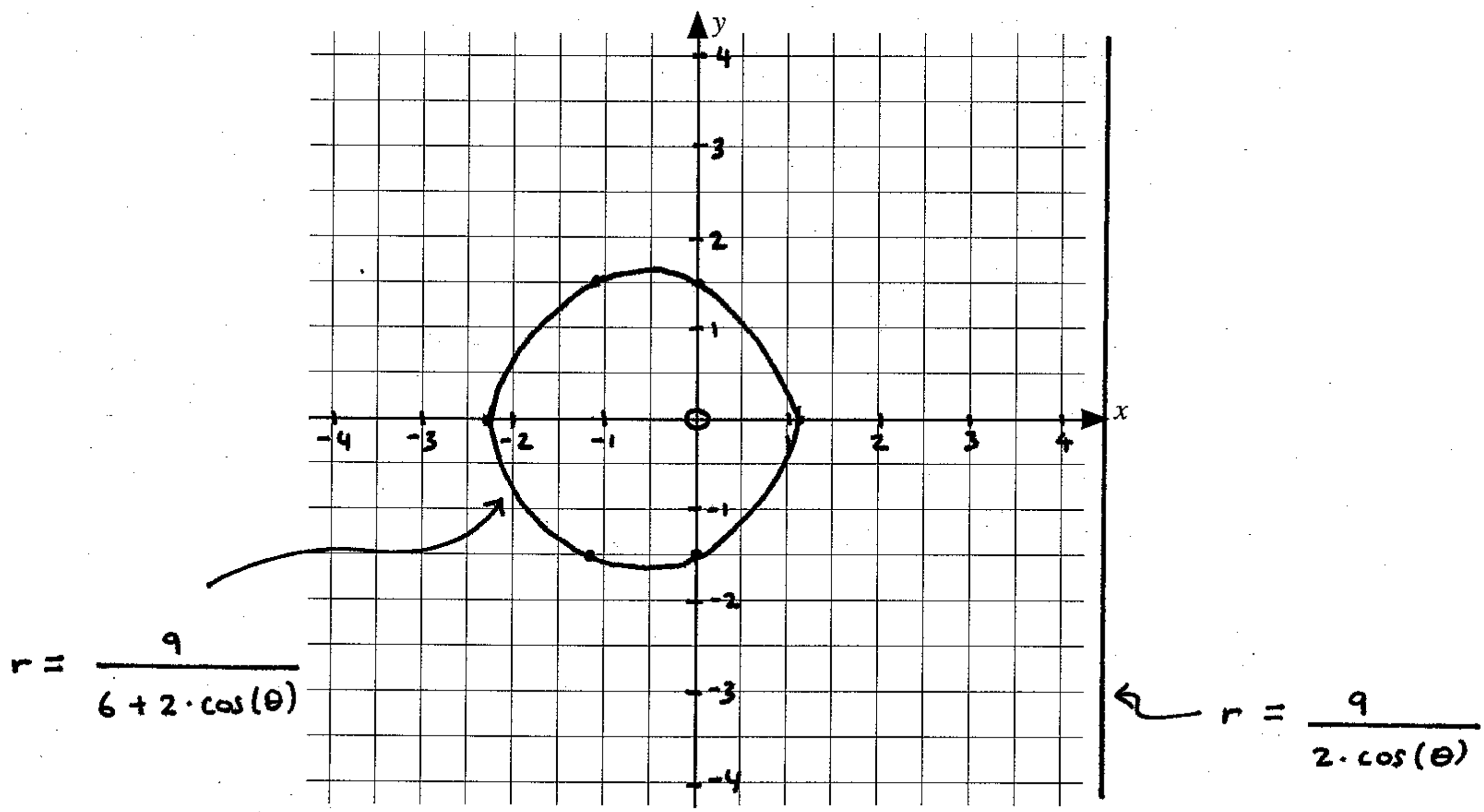
θ	r
0	$9/8$
$\pi/2$	$9/6$
π	$9/4$
$3\pi/2$	$9/6$



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2. Find parametric equations for each of the following lines described below.

(a) The line that passes through the point (–2, 2, 4) and is perpendicular to the plane $2x - y + 5z = 12$.

The direction vector of the line is the normal vector to the plane, $\langle 2, -1, 5 \rangle$.

The vector equation of the line is:

$$\langle x, y, z \rangle = \langle -2, 2, 4 \rangle + t \cdot \langle 2, -1, 5 \rangle.$$

The parametric equations are:

$$x = -2 + 2t$$
$$y = 2 - t$$
$$z = 4 + 5t$$

(b) The line that passes through the two points (4, –1, 2) and (1, 1, 5).

The parametric equations are:

$$x = (1-t) \cdot 4 + t \cdot 1 = 4 - 3t$$

$$y = (1-t)(-1) + t \cdot 1 = -1 + 2t$$

$$z = (1-t)(2) + t \cdot 5 = 2 + 3t.$$

(c) The line that is formed by the intersection of the two planes $x + y - z = 1$ and $2x - 3y + 4z = 5$.

First we must find a point on the line.

Suppose $z = 0$. Then:

$$x + y = 1$$
$$2x - 3y = 5$$

The solution is: $x = \frac{8}{5}$ $y = -3/5$.

The point is: $(8/5, -3/5, 0)$.

The direction vector of the line is the cross product of the two normal vectors.

$$\langle 1, 1, -1 \rangle \times \langle 2, -3, 4 \rangle = \begin{matrix} i & j & k & i & j \\ 1 & 1 & -1 & 1 & 1 \\ 2 & -3 & 4 & 2 & -3 \end{matrix}$$

$$= \langle 1, -6, -5 \rangle.$$

Vector equation of line:

$$\langle x, y, z \rangle = \langle 8/5, -3/5, 0 \rangle + t \cdot \langle 1, -6, -5 \rangle$$

Parametric equations:

$$x = 8/5 + t$$
$$y = -3/5 - 6t$$
$$z = -5t$$

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3. Find the length of each of the following curves.

(a) $x(t) = 3t^2$ and $y(t) = 2t^3$ where $0 \le t \le 2$. $\frac{dx}{dt} = 6t \quad \frac{dy}{dt} = 6t^2$

$$\text{Arc length} = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^2 6t \cdot \sqrt{1 + t^2}\, dt \qquad u = 1 + t^2, \quad \frac{du}{3t} = dt$$

$$= \left[\frac{4}{3} u^{3/2}\right]_1^5$$

$$= 4/3 (5^{3/2} - 1)$$

(b) $r = \frac{1}{\theta}$ where $\pi \le \theta \le 2\pi$. $f(\theta) = 1/\theta \quad f'(\theta) = -1/\theta^2$

$$\text{Arc length} = \int_\pi^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2}\, d\theta$$

$$= \int_\pi^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2}\, d\theta$$

$$= \left[\frac{-\sqrt{\theta^2+1}}{\theta} + \ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_\pi^{2\pi} \quad \text{(using Integration formula.)}$$

$$= \frac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$$

(c) $r = \sin^3\left(\frac{\theta}{3}\right)$ where $0 \le \theta \le \pi$. $f(\theta) = \sin^3(\theta/3)$

$$f'(\theta) = 3 \cdot \sin^2(\theta/3) \cdot \cos(\theta/3) \cdot 1/3$$

$$\text{Arc length} = \int_0^\pi \sqrt{\sin^6(\theta/3) + \sin^4(\theta/3) \cdot \cos^2(\theta/3)}\, d\theta$$

$$= \int_0^\pi \sin^2(\theta/3)\, d\theta$$

$$= \frac{1}{2}\int_0^\pi (1 - \cos(2\theta/3))\, d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{3}{2}\sin(2\theta/3)\right]_0^\pi$$

$$= \pi/2 - 3\sqrt{3}/8.$$

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4. In this problem you may assume that a is a positive constant. Find the coordinates (x and y) of the points where the curve defined by:

$$x(t) = 2a \cdot \cos(t) - a \cdot \cos(2t) \qquad \text{and} \qquad y(t) = 2a \cdot \sin(t) - a \cdot \sin(2t)$$

has **(a)** horizontal, and **(b)** vertical tangent lines. Once you have identified these points, use the axes provided (see next page) to sketch the curve.

$$x'(t) = -2a \cdot \sin(t) + 2a \cdot \sin(2t)$$

$$y'(t) = 2a \cdot \cos(t) - 2a \cdot \cos(2t).$$

<u>Solutions of $x'(t) = 0$:</u>

$$\sin(2t) - \sin(t) = 0$$

$$\sin(t) \cdot \left[2 \cdot \cos(t) - 1\right] = 0$$

$$t = n\pi \quad \text{and} \quad t = \frac{\pi}{3} + 2n\pi, \quad t = \frac{5\pi}{3} + 2n\pi \qquad n \in \mathbb{Z}.$$

<u>Solutions of $y'(t) = 0$:</u>

$$\cos(2t) - \cos(t) = 0$$

$$2\cos^2(t) - \cos(t) - 1 = 0$$

$$\cos(t) = \frac{1 \pm \sqrt{1^2 - 4(2)(-1)}}{(2)(2)} = -\frac{1}{2}, \ 1.$$

$$t = (2n+1)\frac{\pi}{2} \quad \text{and} \quad t = \frac{2\pi}{3} + 2n\pi, \quad t = \frac{4\pi}{3} + 2n\pi \qquad n \in \mathbb{Z}.$$

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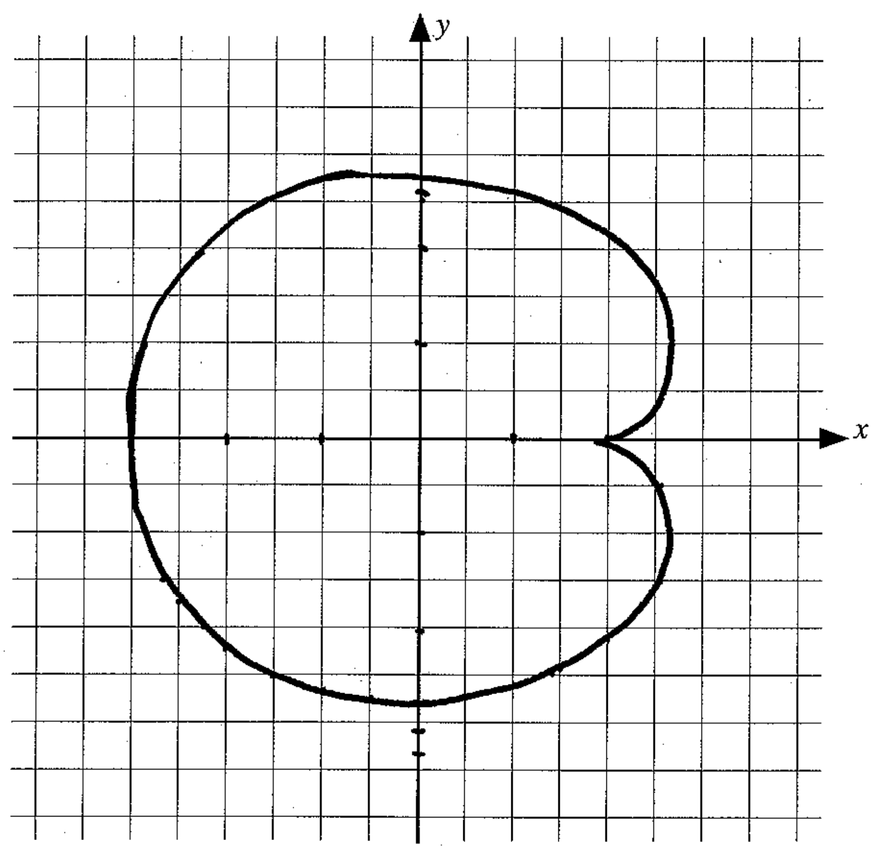
Points where tangent line is vertical:

$$(-3a, 0), \quad (\tfrac{3}{2}a, a), \quad (\tfrac{3}{2}a, -a)$$

Points where tangent line is horizontal:

$$(-a/2, \ 3\sqrt{3}/2 \cdot a), \quad (-a/2, \ -3\sqrt{3}/2 \cdot a)$$



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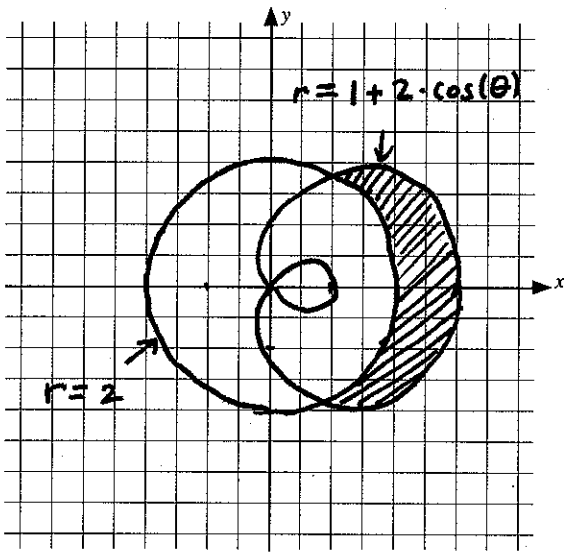
5. Each part of this problem describes a region of the *xy*-plane. Find the area of each region.

(a) The region lies within the curve $r = 1 + 2\cdot\cos(\theta)$ and outside the circle r = [redacted] 2.



To find the limits of integration, find the intersection points of the curve:

$$1 + 2\cdot\cos(\theta) = 2$$

$$\cos(\theta) = 1/2$$

$$\theta = \pm \pi/3.$$

$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(1 + 2\cos(\theta))^2 - 2^2 \right] d\theta$$

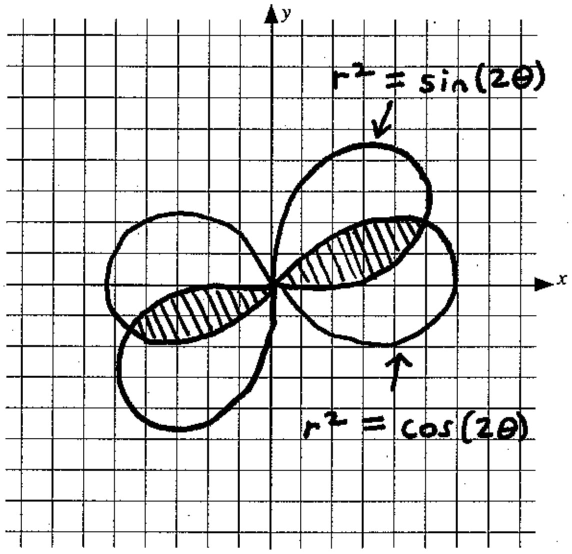
$$= \int_0^{\pi/3} (4\cos(\theta) + 4\cos^2(\theta) - 3)\, d\theta$$

$$= \int_0^{\pi/3} (4\cos(\theta) + 2\cos(2\theta) - 1)\, d\theta$$

$$= \Big[4\sin(\theta) + \sin(2\theta) - \theta \Big]_0^{\pi/3}$$

$$= \frac{15\sqrt{3} - 2\pi}{6}$$

(b) The region lies inside both $r^2 = \cos(2\theta)$ and $r^2 = \sin(2\theta)$.



To find the limits of integration, find the intersection point:

$$\sin(2\theta) = \cos(2\theta)$$
$$2\theta = \pi/4$$
$$\theta = \pi/8.$$

Next, solve the equation: $r^2 = \cos(2\theta) = 0$
$\theta = \pi/4$.

$$\text{Area} = \int_0^{\pi/8} \sin(2\theta)\, d\theta + \int_{\pi/8}^{\pi/4} \cos(2\theta)\, d\theta$$

$$= \left[-\frac{1}{2} \cos(2\theta) \right]_0^{\pi/8} + \left[\frac{1}{2} \sin(2\theta) \right]_{\pi/8}^{\pi/4}$$

$$= \frac{1}{2}\left(2 - \sqrt{2}\right).$$

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6. Find an equation for the [redacted] plane that passes through the points $(1, 0, -1)$ and $(2, 1, 0)$ that is parallel to the line of intersection of the planes $x + y + z = 5$ and $3x - y = 4$.

The direction vector of the line is given by:

$$\langle 1, 1, 1 \rangle \times \langle 3, -1, 0 \rangle = \begin{array}{ccccc} i & j & k & i & j \\ 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & 3 & -1 \end{array}$$

$$= \langle 1, 3, -4 \rangle.$$

The vector $\langle 1, 3, -4 \rangle$ lies in the plane.

A second vector is given by:

$$\langle 2-1, 1-0, 0--1 \rangle = \langle 1, 1, 1 \rangle.$$

The normal vector to the plane is:

$$\langle 1, 3, -4 \rangle \times \langle 1, 1, 1 \rangle = \begin{array}{ccccc} i & j & k & i & j \\ 1 & 3 & -4 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

$$= \langle 7, -5, -2 \rangle.$$

The point $(1, 0, -1)$ lies in the plane. The equation of the plane is:

$$7(x-1) + -5(y-0) + -2(z--1) = 0.$$